For those who understand and enjoy mathematics its symbolism is a gateway to an elegant, satisfying, and powerful mental apparatus. But for those to whom mathematics is a source of difficulty and confusion, these same symbols are more often perceived as barriers to understanding. Those who understand mathematics — who can attach correct mathematical meanings to its symbols — pay little attention to the symbols themselves as they pass beyond them to the associated mathematical ideas. But those who do not understand mathematics do not get beyond its symbols, which rightly or wrongly they regard as one of their main sources of difficulty.

My personal view is that though the power of mathematics resides in its ideas, access to this power is largely dependent on its notation, and that the better the notation the more effectively we can handle the ideas. (Compare the difficulty of multiplying CLXIV by XVIII with the relative ease of multiplying 164 by 18). Even for competent mathematicians, therefore, there is much to recommend the study of notation in its own right; and particularly, what are the properties of, and criteria for, a good notation. And for those concerned with mathematical education, a study of the particular problems of learners with respect to its symbolism would seem to be indispensable if help is to be given in an area where it is particularly needed.

The present collection of papers is offered as a contribution in this area, together with the hope that others too may begin to perceive mathematical symbolism as a subject likely to reward greater study than it has yet received. *R.R.S.*

Difficulties with Mathematical Symbolism: Synonymy and Homonymy

Josette Adda

We know that the confusion between meaning and sign (in French: *signifié/signifiant*) is the root of a great number of mistakes in mathematics. Particularly, instead of making easier the approach to the mathematical concept represented, the sight of the design often produces a disturbance to understanding; it leads to mistaking the *drawing* for the presented *idea,* as idolatrous people do. I will demonstrate — by presenting many genuine examples which I have met in mathematical classrooms at every level — the mathematical roles of synonymy and homonymy.

Mathematical objects are, by nature, abstract objects. Only through their denotations is it possible to encounter them. So, the problem of the linguistic relation of meaning (i.e., the relation between *signified* and *signifier*) is particularly crucial for mathematical understanding. Teaching and learning situations bring to light difficulties inherent to mathematics. Failures by students are signs of epistemological obstacles. So we are going to study our problem through paradigmatic cases, observed during mathematics classes.

Thinking of the role of symbols, we would be happy to have a one-to-one correspondence: SYMBOL↔MATHEMATICAL OBJECT. We would like: (1) that each symbol should denote one mathematical object and only one, and this not only for the teacher (or writer of a textbook or an examination) but also for each of the students — and the same one for everybody! (2) that each mathematical object be represented by one single symbol. Alas! It does not work like this (see, for instance, Skemp 1971, Freudenthal 1973, and Adda 1975-1976); so we will see that we cannot escape the linguistic problems of synonymy and homonymy.

I SYNONYMY

Synonymy of symbols is decisively related to the problem of identity. One never needs to say that one object is itself, but one often says that two objects are only one (for instance: "the *two* numbers a and b are equal so that they are the *same* number"). This is a frequent use, but it is a misuse because what we intend to mean is that the two *names* are names of the same single object, that they are synonymous.

Visible Language, XVI 3 (Summer 1982), pp. 205-214.
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0022-2224/82/007-0205$02.00/0 © 1982 Visible Language, Box 1972 CMA, Cleveland, OH 44106.

The symbol "=" is of constant use in mathematics and its function is to mean that the symbols written on its left and its right denote the same object. Thus 2 = 2,00 = 4/2 and this means that "2" and "2,00" and "4/2" are symbols referring to the same object, the same number. [Editor's note: As readers will know, the decimal point is rendered in French usage by a comma.] But many studies (e.g., Kangomba 1980) show that pupils, and even some teachers, often say that 2 is a natural number but neither a decimal nor a rational number, while 2,00 is decimal but neither natural nor rational and 4/2 is rational but neither decimal nor natural![1] If we refuse, as some people do, the complete identification by embedding the set of natural numbers into a part of the set of rational numbers we have to renounce definitely to write "4/2 = 2" which is very useful!

Being unaware of the synonymy, pupils can write without qualms: "2 + 3 = 5 + 7 = 12 × 2 = 24". This comes from a general use in school of questioning statements such as "2 + 3 = . . .", in which the symbol "=" does not have the same meaning as "equals" but rather that of "gives as result," and so the above statement can be understood as a sequence of manipulations on a calculator. Writing in the same way as when operating with a calculator is a sensible behavior but, unfortunately, it does not lead to correct mathematical statements.

Brookes (1980) notices that whatever they have been taught about this, when asked: "Look at '7 + 8 = 14'; correct the mistake, please" almost everybody has the same initial reaction and puts "15" at the place of "14"; far less spontaneous are other corrections such as putting "7 + 7" in place of "7 + 8", or even "8 + 8 = 16". . . . This shows that the asymmetric meaning of "=" is pregnant for all of us. Furthermore, while the mathematician has emphasized that 2 and 2,0 are the same object in some later physics lesson the pupil will be told of a crucial distinction between 2 and 2,0 (about accuracy).

Many teachers and textbooks authors are disturbing. F. Cerquetti (1981) quotes a strange mathematics textbook in which, in an exercise, "2,10" is described as "incorrect writing":

8. Supprime les zéros inutiles:

écriture correcte	*écriture incorrecte*
2,1	2,10
. . . .	04,05
. . . .	30,100
. . . .	108,20
. . . .	0,00050
. . . .	1,2800
. . . .	104,0

Though, three pages down, fortunately, one can see the expression "2,50f" in another exercise!

II HOMONYMY

When two different objects have the same (or nearly the same) designation, problems of understanding are bound to arise, and we know of cases in which designations differ in spelling only and in which children, listening to a text which does not make sense for them, mispell it. It seems that children who have the greatest difficulties with the linguistic problem of spelling are also those who are unaware of its function; it would be fruitful to enable them to become aware of the importance of the convention.

As an example, an eleven-year-old French boy noted for his very poor language performances (especially in spelling) was, on the contrary, very bright when working with LOGO (Papert's computer display turtle). He decided to draw on the screen a camera, the program of which he called FOTO. But the drawing appearing on the screen was not satisfying, so he prepared a new program and called it FAUTAU, and after this a new one called FAUTEAU, and another, the good one, called FAUTTEAU. Most surprising is that this boy never did confuse the names when he was typing on the fingerboard. The spelling conventions decided by himself were very coercive for him. In mathematical language we often use the same (or quite similar) notations for different concepts and this creates difficulties unless the difference is suitably emphasized.

1 Confusion between similar notations

I shall precise this type of confusion by presenting examples about the symbolism of a sequence of figures followed by a comma and of a sequence of figures.

Emmanuelle (an autistic girl, ten years old, studying in a special class for mentally handicapped children) looks at the three exercises written on the blackboard by the teacher. In each of them two or three decimal numbers occur, some of them being written with commas (remember that this is the French use). In her exercise-book, she writes operations where not only the numbers appear to be combined as at random, but also the decimal notation is often decomposed and treated as a symbol denoting a couple of numbers.[2]

Well, will you say, this is a very extreme case! But what about the comparison between 5,2 and 5,18? Many experiments (e.g., inquiries by the IREMS of Rouen and Strasbourg) show that even fifteen-year-old pupils claim that 5,2 < 5,18. Is not it provoked by the same confusion as Emmanuelle's? "5,2" is not seen here as another name for the number also written "5,20". Instead there is on one side the "5" and on the other the "2" which means less than "18"! In the opposite direction, we can find situations in which a couple of natural numbers is confused with a decimal one. For instance, I saw a 13-year-old pupil faced with the definition of integers as classes of couples of natural numbers* so that he had to perform additions of couples of naturals as an introduction to additions of integers. I observed on his paper the following mistake:

$$\begin{array}{r} (4,7) \\ +\ (3,5) \\ \hline =(8,2) \end{array} \quad \text{instead of} \quad \begin{array}{r} (4,7) \\ +\ (3,5) \\ \hline =(7,12) \end{array}$$

This can be compared with the following line that G. Glaeser saw in an examination of complex numbers (at first year of university):

$$\frac{1+i}{1-i} = \frac{(1,0)+(0,1)}{(1,0)-(0,1)} = \frac{1,1}{0,9} = 1,22$$

So even with the plain problem of figures with comma — with or without parentheses (a very small difference?) — we can see those confusions at many various levels of studies.

Another type of example of confusion between similar notations involves the case of notation by *nothing,* i.e., juxtaposition. Many pupils are troubled by its use for products, and so, being ignorant of the rules for algebraic writing, they are led to the following "simplification" well known by all teachers:

$$\frac{\cancel{6}}{3a\cancel{6}} = \frac{1}{3a}$$

2 Confusion between different linguistic levels

In section 1 the confusion was only by pupils on their own: they identified expressions which were not exactly identical. But now we will consider confusions in which teachers share the responsibility because of language misuses.

*For those unfamiliar with this definition of integerns, we recommend that this example be omitted. Eds.

Logicians emphasize the need to use the symbolism of quotation marks to distinguish the symbol, taken as object, from its referent. Actually, in writing, quotation marks are often omitted, and for oral discourse it is quite difficult to make them perceptible. One often misuses *referent* when one intends to speak about the *symbol*: For instance, consider the sentence, "An even number ends with 0,2,4,6, or 8." Here is the same confusion of linguistic levels as in the sentence: "Paris has five letters." Indeed, nobody is disturbed by the confusion between Paris which is a city and "Paris" which is the name of a city (here is a mine for jokes and puns from the most ordinary kind to Lacan's), nor by the confusion between the even number which is divisible by two and its symbol in decimal notation. But when we listen to a teacher in primary school, we are very surprised by the abuses made during the study of numeration in which both numbers and their denotations are considered. We often hear (and even read in textbooks) the following sentence: "To multiply a number by ten, add a zero." In this strange formulation, multiplication is an operation on numbers while addition is intended as an operation on sequences of figures (i.e., a metamathematical operation) and nobody tells the pupils that! So do not be surprised to hear some poor child saying: "2 + 1 are 21."

Confusions are frequent in the study of fractions. For instance, 1/2 and 2/4 are equal, but "1/2" has a prime denominator while "2/4" has not. If you say or write the previous sentence without quotation marks (as it is generally done), that will be quite disturbing for the meaning of equality.

In these examples, we have seen two levels confused in the same discourse. Even when there is only a single level, we can find misunderstanding in communication between teacher and pupil if one of them thinks at one linguistic level and the other one at another. The date with "1979" being written on the blackboard, I observed a teacher asking a nine-year-old (in a special class for the mentally handicapped) to write a larger number, the pupil wrote in the middle of the blackboard a very large: 352

In some sense he was right, but I had to convince the teacher!

Jaulin-Mannoni (1975) asked a child in front of this drawing

to draw "three times a" and he drew

a a a

and not three drawings of a tree as she expected. That certainly does not show a difficulty about multiplication but rather about the confusion between symbol and reference.

III THE PROBLEM OF VARIABLES

The situation of homonymy — in which a symbol is considered as meaning, at the same time, both its referent and the symbol itself — often occurs, as in the last example, with the use of letters for symbols of variables. For non-mathematicians this use is particularly disturbing. Actually, we are dealing here with the main specificity of mathematical language and, for people who failed in mathematical learning, the language was often the barrier where they got stopped. Baruk (1977) asked Daniel what an equation is; he answered, "It is figures and letters." We can hear others saying, "Oh mathematics! Some a, some b, and x, and . . . equals zero." This is the only mark left by ten years of mathematics in much of the population.

This use of letters is an important difficulty inherent to mathe-
matics. We cannot avoid it, but perhaps we can make it more explicit
to pupils. We are simultaneously confronted with phenomema of homo-
nymy and synonymy: apparent homonymy between the symbol and the signified (but we will see later on that some perverse exercises are based on it) and hidden synonymy between the letter and other designations of an object.

For instance, in "$2x + 3 = 0$", "x" is synonymous of "$-3/2$"; in (1) "$ax^2 + bx + c = 0$", the symbol "x" is synonymous of the two in "$-b \pm \sqrt{b^2 - 4ac}/2a$", "$c$" is synonymous of "$-ax^2 - bx$", and "$ax^2 + bx + c$" is synonymous of "0".[3] But all the letters have not here the same function: for instance, in an equation some letters represent *unknowns* and others represent *parameters,* depending on the role in the problems. Example: (1) is a second degree equation in x, or a first degree equation in c.

Furthermore, the meaning of a mathematical sentence depends on the structure which interprets the symbols, and each letter denotes any object of the reference set attributed to this letter. For instance, consider the expression:

"$\exists$ x ($x + 3 = 1$)"

if "x" ranges over $\mathbb{N}$ (the set of natural numbers) the sentence is false, and if "x" ranges over $\mathbb{Z}$ (the set of positive and negative whole numbers) it is true, while

$\forall x \; (x + 3 = 1)$

(1)

is false over $\mathbb{N}$ and over $\mathbb{Z}$ and

$\forall x \, R(x + 3, 1)$

(2)

is true over $\mathbb{N}$ if "R" means the ordinary relation of order (so that the structure $<\mathbb{N}, \geqslant>$ is said to be a "model" of the sentence [2]), and not true if "R" means the relation of equality, etc. . . . .

Davidov and Wilenkin (see Freudenthal 1974) conducted experiments in teaching the use of variables at the very beginning of mathematical studies, in elementary school. Pupils then seem to be more able to understand the meaning of operations; they are not distracted by computational difficulties when confronted with a word-problem with letters. It seems to be easier to teach them to substitute the writing of numbers to letters than to teach them, as we generally do, to generalize by letters the writing of numbers. For instance, when we say, "Let n be a number," we often hear children protest: "n is not a number, it is a letter." A good process is (as Varga does) to tell them "choose a number" (and each pupil chooses his own) and "do so and so . . . ." Then we explain what was done by each one by saying "the number," and quickly it becomes more comfortable to abbreviate and more natural to say "n" for "the number." Note that this process only works when they can forget that "n" is a letter.

So we have to stand up against the perverse exercises in which two linguistic levels are occurring: such as asking children this strange question: "What is the set of the x such that x is a vowel?" which mixes the letters considered as objects and the letters considered as symbols for variables. Do not be surprised if pupils are troubled and answer "$\emptyset$"! The convention to designate variables by letters was taken for mathematics about numbers not about letters (just to avoid designation of a number by another number!) Still worse are traps such as this classical one: "Calculate $(x - a)(x - b)(x - c)$ . . . . $(x - z)$," in which two levels are deliberately mixed. This could be a pleasant joke[4] but please be aware that answering "0" is not compatible with the mathematical use of variables. You have to think to the factor $(x - x)$ where, in a true mathematical situation, you would have to consider the difference between two numbers (independently of the alphabet of the country!).

Homonymy is also dangerously encountered in situations where we use the same letters for names of objects and for linked variables.[5] For instance, I observed a teacher referring to a linear map as "f" and following exercises with various special cases dealing with maps always called "f" as well, asking whether each f was linear or not. Pupils were completely disturbed, trying reasoning involving vicious circles (the property of the f being intended as given in the definition).

Points as variable symbols

The use of "etc. . . . " and of ". . . " can be connected with the use of variables. We encounter them with an infinity of references, and the authors hope that we shall discover the meaning through the context (see Adda 1979). Actually, questions such as "Complete 1,3,5,7,. . . " (though of frequent use by psychologists) are not mathematical because, mathematically, here ". . . " can mean anything. But pupils have to understand many expressions like:

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

and the canonical interpretation of them can be difficult when pupils are not familiar with the context.

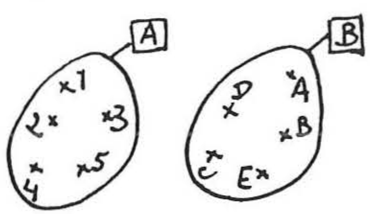
We find another use of points in "fill-the-blanks" exercises. For instance:

$$\begin{array}{r} 4\,. \\ +\ .5 \\ \hline =71 \end{array}$$

I think this is worse than letters because here the same symbol is used in cases where one would have written different letters.

BY WAY OF SYNTHESIS

To conclude, it might be interesting to examine the answer of a twelve-year-old boy to a teacher who had asked for an example of two disjoint sets:



Curiously, the teacher found it good! For me, this is a mathematical monster. Not only could the sets have a non-empty intersection and even be equal in many cases (for instance $A = 1, B = 2, C = 3, D = 4, E = 5!$), but let us not forget that necessarily $A = A$ and, even more, $B = B$! Is it not that the reason this child did not see this as a special monster is because for him, as for many people in the non-mathematical world, mathematical symbolism is considered as a sorcerer's code for which ordinary people cannot hope to discover a key?[6]

1. It seems to me that a large part of the responsibility lies in textbooks and in such teachers' expressions as: "A decimal number is a number *written* as . . . " and "A rational number is a number *written* as . . . " instead of "a decimal [resp. a rational] number is a number which *can be written* as . . . ."
2. Sometimes figures were recombined in other numbers so that, one day, I observed:

On the blackboard	On Emmanuelle's paper
13,5 − 9,5	$\begin{array}{r} -135 \\ 59 \\ 1 \\ \hline 164 \end{array}$

3. Of course it is not simple. Here is the situation of synonymy generated by *descriptions*. It is very complex. In the beginning of our century logicians with Bertrand Russell thought very much about paradoxes such as: "Walter Scott is the author of *Waverley*" so that "Walter Scott" is synonymous of "the author of *Waverley*," and in nearly all situations you can write one for the other — but not in the descriptive sentence itself, obtaining "Walter Scott is Walter Scott" which is a very different sentence!
4. One French textbook (by IREM of Strasbourg) presents it as a "poisson d'Avril" but not all teachers are so honest!
5. Also, though I did not observe cases of misunderstanding, we can note the use of the same letter for names of special objects and for variables with, for instance, "π" (taken as for the special number and as for names of planes . . .) or "i" (with $i = \sqrt{-1}$ or names of integers, . . .) etc. . . .
6. I am grateful to C. Berdonneau for many corrections of the poor English of my first version of this paper.

All the quoted examples are French. Those about commas are probably avoided by the notation of the decimal point. I am not able to know if there exists a similar perturbation for English and American pupils.

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Emotional Responses to Symbolism

Laurie G. Buxton

Special difficulties often arise in reading mathematics because of the symbols and notation that are used. This is caused not only by the range of symbols and their density of meaning (interiority) but also by strong emotional responses raised by certain symbols or combinations. These feelings may reflect unpleasant memories of when the symbols were first encountered, but may even derive from an unease with the shape of some of them.

Much learning hinges upon the decoding of symbols, for it is mainly by means of written symbols that the knowledge the human race has accumulated is stored. Most of us learn satisfactorily to read our own language, though any of us can be confronted with passages of prose or poetry which we are able to translate from the written symbols to the spoken word, but cannot claim readily to comprehend. On the whole we remain comfortable when presented with a piece of our own written language whose symbols do not, with some reservations discussed below, occasion us disquiet. However, how are they regarded by someone who has not been able to learn to read? The range of unpleasant feelings is considerable. The mere sight of symbols of language will occasion fear, distaste, embarrassment, and shame. Anyone who has sought to teach an adult illiterate will confirm that this statement is not too strong. There is, in fact, a vicious circle whereby the emotional response to the symbols is such as to inhibit the individual's cognitive processes, which may in themselves be perfectly adequate to acquire the skill of reading. It is difficult to put oneself in the position of a non-reader (or even a pre-reader, though we have all passed through this stage). But once we introduce mathematical symbols, most of the population can be put precisely in this situation.

I shall describe three experiments on reactions to symbols, and hope then to offer explanations of two of them. The first was conducted with various groups of people most of whom knew some mathematics and had a generally positive attitude to the subject. The following statement was shown on a screen by an overhead projector:

$\phi(x)$ is continuous for $x = \xi$ if, given δ, $\exists$
$\varepsilon(\delta)$ s.t. $|\phi(x) - \phi(\xi)| < \delta$ if $0 \leqslant |x - \xi| \leqslant \varepsilon(\delta)$

Visible Language, XVI 3 (Summer 1982), pp. 215-220.
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0022-2224/82/007-0215$02.00/0