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Symbols, Icons, and Mathematical Understanding

William Higginson

Extracts are taken from the biographies of Hobbes, Rousseau, Darwin, and Russell which refer to their mathematical education. The common feature of an attraction toward geometry and an aversion to elementary algebra is noted. These experiences are analysed using theoretical positions promulgated by Davis, Hersh, Skemp, and Bruner. The central thesis is that these men probably have had difficulty learning elementary algebra because they had failed to develop a strong image or iconic representation of the concepts involved. This thesis is developed in relation to "squaring a binomial," the concept which troubled both Rousseau and Russell.

> *Mathematics is often considered a difficult and mysterious science, because of the numerous symbols which it employs.*
> *A.N. Whitehead*

Much of the power of mathematics stems from the potency of its symbols. There is, however, a price to be paid for this potency. The symbols which serve as highly effective tools for some are the most formidable of barriers for others. In the following pages a thesis is outlined which attempts to account for some of the difficulties which learners meet when studying mathematics. The method of approach is largely biographical; the essence of the argument: that we have paid too little attention to the role of images in mathematical understanding.

The unique cluster of insights, associations, and emotions which characterizes every encounter of individual with idea is never easy to capture. One of the few sources to which we can turn in such a quest is biographical literature. The examination of this literature for accounts of man meeting mathematics reveals some interesting commonalities in the experiences of a number of people. For our purposes we consider four distinguished thinkers; Thomas Hobbes (1588-1679), Jean-Jacques Rousseau (1712-1778), Charles Darwin (1809-1882), and Bertrand Russell (1872-1970).

One of the most striking features of John Aubrey's marvelous collection of short biographies, *Brief Lives,* is the picture it gives of the impact of the release of the mathematical sciences from the Greek and

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Latin tongues. Henry Gellibrand is described as, "good for little a great while, till at last it happened accidently, that he heard a Geometrie Lecture," and for Richard Stokes, M. D., we find, "His father was Fellow of Eaton College. He was bred there and at King's College. Scholar to Mr. W. Oughtred for Mathematiques (Algebra). He made himself mad with it, but became sober again, but I fear like a crackt-glasse.... Became a Sott." Few entries can compare for vividness, however, to the one for the philosopher Hobbes where his first exposure to geometry is noted.

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 El. libri I. He read the Proposition. By G–, sayd he, (he would now and then sweare an emphaticall Oath by way of emphasis) this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.
>
> I have heard Mr. Hobbes say that he was wont to draw lines on his thigh and on the sheetes, abed, and also multiply and divide. (p. 309)

Mathematical ideas play a very limited role in Rousseau's spirited autobiographical *Confessions*. There are, however, two passages which are of interest. The first perhaps tells us something of eighteenth century attitudes about mathematicians. In 1744 Rousseau had a short and less than satisfactory liason with a sultry Venetian courtesan called Giulietta. At their parting the vengeful young woman advised Rousseau in a cold and scornful voice to "Give up the ladies and study mathematics" (p. 302). The other passage is more appropriate for our purposes for in it Rousseau describes some of his mathematical education as follows:

> I have never been sufficiently advanced really to understand the application of algebra to geometry. I disliked that way of working without seeing what one is doing; solving a geometrical problem by equations seemed to me like playing a tune by turning a handle. The first time I found by calculation that the square of a binomial figure was composed of the square of each of its parts added to twice the product of one by the other, despite the fact that my multiplication was right I was unable to trust it until I had drawn the figure on paper. It was not that I had not a great liking for algebra, considered as an abstract subject; but when it was applied to the measuring of space, I wanted to see the operation in graphic form; otherwise I could not understand it at all. (p. 3)

Charles Darwin's massive contribution to the intellectual life of the nineteenth and twentieth centuries would most likely have come as a considerable surprise to his youthful contemporaries, for his scholastic record as a schoolboy and undergraduate was far from prepossessing. In 1847, sixteen years after receiving his B. A. from Cambridge, Darwin wrote in a letter to his friend Hooker, "I am glad you like my *Alma Mater,* which I despise heartily as a place of education." In another passage in his autobiography we find:

> During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. (p. 18)

Darwin's son Francis, who edited the *Autobiography and Selected Letters* of his father, notes in another passage:

> My father's letters to Fox show how sorely oppressed he felt by the reading for an examination. His despair over mathematics must have been profound, when he expresses a hope that Fox's silence is due to "your being ten fathoms deep in the Mathematics; and if you are, God help you, for so am I, only with this difference, I stick fast in the mud at the bottom, and there I shall remain." Mr. Herbert says: "He had, I imagine, no natural turn for mathematics, and he gave up his mathematical reading before he had mastered the first part of algebra, having had a special quarrel with Surds and the Binomial Theorem. (p. 114)

It is perhaps the passion which most catches one's attention in these passages. One might well expect, for instance, that men whose contributions were to non-mathematical fields should have had certain difficulties with the discipline. It is, however, something more of a surprise to find that one of the finest logico-mathematical minds of the last hundred years experienced difficulties with elementary algebra almost identical to those of Rousseau. In his autobiography (1968) Bertrand Russell writes:

> The beginning of Algebra I found far more difficult, perhaps as a result of bad teaching. I was made to learn by heart: "The square of the sum of two numbers is equal to the sum of their squares increased by twice their product." I had not the vaguest idea what

this meant, and when I could not remember the words, my tutor threw the book at my head, which did not stimulate my intellect in any way. (p. 34)

This difficulty was to prove a temporary one: "After the beginning of Algebra, however, everything else went smoothly" (p. 34). And some seventeen years later Russell would be embarking on what has been called "the longest chain of deductive reasoning that has ever been forged" (*Spectator,* p. 142). This, of course, was *Principa Mathematica,* the three-volume treatise on the foundations of mathematics which Russell co-authored with Alfred North Whitehead. The anonymous reviewer in the *Spectator* wrote of a work which "may be said to mark an epoch in the history of speculative thought" but went on to observe:

> It is easy to picture the dismay of the innocent person who out of curiosity looked into the later part of the book. He would come upon whole pages without a single word of English below the headline; he would see instead, scattered in wild profusion, disconnected Greek and Roman letters of every size interspersed with brackets and dots and inverted commas, with arrows and exclamation marks standing on their heads, and with even more fantastic signs for which he would with difficulty so much as find names. (p. 142)

We wish to set our analysis of these biographical excerpts in the context of the view of four theoreticians: the mathematicians Davis and Hersh on the role of symbols in mathematics, the mathematics educator Skemp on types of mathematical understanding, and the cognitive psychologist Bruner on modes of symbolic representation.

In their recent book Davis and Hersh (1980) have a section entitled "Symbols" in which they observe:

> What do we do with symbols? How do we act or react upon seeing them? We respond in one way to a road sign on a highway, in another way to an advertising sign offering a hamburger, in still other ways to good-luck symbols or religious icons. We act on mathematical symbols in two very different ways: we calculate with them, and we interpret them.
>
> In a calculation a string of mathematical symbols is processed according to a standardized set of agreements and converted into another string of symbols. This may be done by machine: if it is done by hand, it should in principle be verifiable by a machine.
>
> Interpreting a symbol is to associate it with some concept or mental image, to assimilate it to human consciousness. The rules for calculating should be as precise as the operation of a computing machine: the rules for interpretation cannot be any more precise than the communication of ideas among humans. (p. 121)

Skemp (1976) has made a distinction between "instrumental" and "relational" understanding which has proven to be useful in analysing situations in mathematics education. In an instrumental approach to the teaching of mathematics major emphasis is placed on the acceptance and application of definitions and rules. The questions of why one would want such definitions and how the particular rules come into being are not appropriate in the instrumental approach. They are, however, essential features of the relational approach.

One of the most complete theories about the nature of the symbolizing process is the one developed by Jerome Bruner and his co-workers over a number of years (1966, 1968, 1973). Bruner (1973) distinguishes three modes of representation — the enactive, the iconic, and the symbolic:

> Their appearance in the life of the child is in that order, each depending on the previous one for its development, yet all of them remaining more or less intact through life.... By enactive representation I mean a mode of representing past events through appropriate motor response.... Iconic representation summarizes events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images. Images stand for perceptual events in the close but conventionally selective way that a picture stands for the object pictured. Finally, a symbol system represents things by design features that include remoteness and arbitrariness. A word neither points directly to its referent here and now, nor does it resemble it as a picture. (p. 328)

An analysis of the biographical statements of the four individuals in question reveals two strong common underlying themes. The first is that of an attraction, often a passionate one, for geometry. We have noted Hobbes' addiction, and Russell's feelings were no less strong. He writes, for instance, "At the age of eleven, I began Euclid with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world" (1968, p. 33).

Even the algebraphobic Darwin had enjoyed geometry. "Again in my last year I worked with some earnestness for my final degree of B. A., and brushed up my Classics, together with a little Algebra and Euclid which latter gave me much pleasure as it did at school" (1958, p. 19).

Rousseau too was attracted to geometry, but he makes a significant qualification. "I went on from there to elementary geometry.... I did not like Euclid, who is more concerned with a series of proofs than with a chain of ideas; I preferred the geometry of Father Lamy, who from that time became one of my favourite authors, and whose works I still re-read with pleasure" (1970, p. 226).

The second commonality is a strong dislike for situations in elementary algebra where it proved difficult to attach any meaning or imagery to the manipulation of symbols. We have seen the views of Rousseau, Russell, and Darwin. Aubrey writes of Hobbes in this connection: "He would often complain that Algebra (though of great use) was too much admired, and so followed after, that it made men not contemplate and consider so much the nature and power of Lines" (p. 309).

With the theoretical positions sketched previously in mind we can set the experiences of Hobbes, Rousseau, Darwin, and Russell in a somewhat more general context. What we have is perhaps not so much a difference between algebra and geometry as branches of mathematics, as rather a situation where the algebra is learned instrumentally, and the geometry relationally. The nature of geometry is such that it lends itself easily to the production of images. This is not so clearly the case with algebra. It is possible, of course, (note the case of Rousseau) to teach geometry instrumentally as well, with the same negative results.

To recapitulate: we see in these four cases, examples of what is probably a very common phenomenon, the presentation of mathematical ideas almost entirely in the symbolic mode of representation. The result of this is that learners fail to have any significant understanding of the situation. Equivalently, using terms which accentuate the iconic nature of their difficulties, they lack *insight* or fail to *see* what is going on. The possibility of remedying this situation by consciously constructing icons for mathematical symbols is an obvious one.

We consider as an example the Rousseau/Russell problem of $(a + b)^2$, the squaring of a binominal. From their descriptions it would seem that both men were encouraged to learn this concept instrumentally. The "rule" is that "the square of the sum of the numbers is the sum of their squares increased by twice their product," that is, $(a + b)^2 = a^2 + b^2 + 2ab$. (An inventive mind like Russell's was able to put even the most arcane bits of knowledge to use. Further on in his autobiography we find, "I used, when excited, to calm myself by reciting the three factors of $a^3 + b^3 + c^3 - 3abc$; I must revert to this practice. I find it more effective than thoughts of the Ice Age or the goodness of God" [1970, p. 38].) Expressed only in this way, this mathematical idea seems little better than a variation on tongue twisters of the genre "Peter Piper picked a peck of pickled peppers."

or more $12 \times 12 = (10 + 2)(10 + 2) = 100 + 20 + 20 + 4 = 144$
generally $12 \times 12 = (11 + 1)(6 + 6) = 66 + 66 + 6 + 6 = 144$
and $8 \times 16 = (2 + 6)(13 + 3) = 26 + 6 + 78 + 18 = 128$

Figure 1.

Yet this need not be, since this is nothing more in some senses than a compact way of noting an infinite number of arithmetic statements of the sort listed in Figure 1; this fact seems seldom to be mentioned in textbooks on elementary algebra.

An even more surprising omission from these books is an obvious iconic representation for the symbolic statement. (It is almost certainly the same one which gained Rousseau's trust.) This icon hinges on the fact, so central to Greek mathematics, that just as we can associate the sum of two positive integers with a particular line segment, we can associate their product with the area of a rectangular figure. If we have two positive integers a and b, a powerful image of the square of their sum is a square of dimension (a + b). As can be seen from Figure 2, this large square is composed of four rectangles: a square of side a, a square of side b, and two identical rectangles of area ab.

Once opened, this door leads to many other related mathematical ideas; for example: the square of a trinomial $(a + b + c)^2$; the cube of a binomial $(a + b)^3$; the difference of two squares $(a^2 - b^2)$; Russell's factoring problem; the square root of two, i.e., find b so that $(1 + b)^2 = 2$. It should probably be noted, as well, that this area is a critically important one in mathematics. It took the genius of a Newton to fully generalize this situation to the case of $(a + b)^n$ and this result was a key step in the development of the calculus. It is not reassuring to observe that we have in many ways progressed very little from the time of Rousseau as far as the teaching of this concept is concerned.

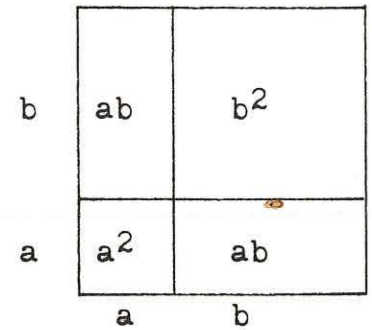

Figure 2.

$$(a+b)^2 = (a+b)\,(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

F O I L

Figure 3. The expansion of a binomial by the FOIL law. The product of the sums is equal to the sum of the products of the first, outside, inside and last terms.

The most popular technique for explaining the product of two binomials in many parts of North America at present is the "FOIL Rule," a blatant appeal to authority (see Figure 3). (There are those who would contend that the whole purpose of teaching mathematics in schools is to have children learn how to accept authority, very often in forms which seem irrational, meaningless, and arbitrary. However, that is another, albeit very important, issue.)

But how typical is our example? Is it possible that only some mathematical concepts have iconic representations or can one legitimately expect to find images playing an important role in the development of all mathematical ideas? It must be acknowledged that the idea of a 'mental image' is one that has been hotly debated in a number of disciplines over the years. Philosophers, psychologists, artists, and mathematicians have all, at one time or another, participated in the fray (Arnheim 1972, Gombrich 1959, Hannay 1971, Mason 1980, Paivio 1971, Plato, and Wertheimer 1968) and there are few principles in the area which would gather universal acceptance. Hadamard's classic work in the field (1954) would seem to show unequivocably that imagery is of critical importance in the thought of creative mathematicians. He quotes Einstein, for instance, as saying "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined" (p. 142). More recently we have the report of the mathematician Papert (1980) who tells of the significant role played in his intellectual development by the image of gears. In general, however, as Bruner (1973) notes, the situation is that "we know little about the conditions necessary for the growth of imagery and iconic representation" (p. 329).

Any conception of mathematical understanding which emphasizes the iconic representation of concepts must acknowledge its roots in the thought of the ancient Greeks. (In his "On Memory and Recollection," Aristotle contended that it was "impossible even to think without a mental picture"). Iverson (1972, 1980) and Hammersley (1979) have written about the limitations of our contemporary mathematical symbols and have made suggestions as to where improvements might be made. Looking to the future it seems obvious that there is great potential for the graphic representation of mathematical ideas through the medium of computers (Papert 1980). It remains to be seen whether or not we will be able to make mathematical symbols more understandable with the aid of computers. In the meantime it is of interest to note that 190 years ago, Samuel Taylor Coleridge, then seventeen years old, observed in a letter to his brother George:

> I have often been surprized, that Mathematics, the Quintessence of Truth, should have found admirers so few and so languid— Frequent consideration and minute scrutiny have at length unravelled the cause—Viz—That, though Reason is feasted, Imagination is starved: whilst Reason is luxuriating in it's proper Paradise, Imagination is wearily travelling over a dreary desart. (p. 7)

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Towards Recording

Nick James and John Mason

Behind the formal symbols of mathematics there lies a wealth of experience which provides meaning for those symbols. Attempts to rush students into symbols impoverishes the background experience and leads to trouble later. In conjunction with manipulating objects it is essential to provide time for talking about their activities and developing their own informal records before meeting the formal symbols of adult mathematicians. We present three examples of children's work which demonstrate these steps in the struggle to move towards recording perceived patterns.

To most people the formal symbols used in mathematics seem cold and lifeless. Even the ubiquitous x is literally an unknown quantity with little meaning. Mathematicians often seem content to lend credence to this view by talking about mathematics as a formal game. This view of mathematical symbols is misleading in its incompleteness. In fact, for symbols of any kind to be of value there must be a wealth of background experience which can be called upon. This article is concerned with developing that rich background experience in the important phase of mathematical learning/investigation which we call TOWARDS RECORDING. The struggle to capture an insight which is as yet pre-articulate is often overlooked in a rush to lead students into formal symbols, resulting in an impoverished if not empty background experience, and producing frustration, anxiety, and math-phobia. We present three examples of childrens' work when time was taken to let the children participate in the struggle towards recording. The results reveal something of the stages in that process.

Keith and Ranjit (age 12½)

Task A Choose a rod and make up the equivalent length using repeating patterns (an example was given). Figure 1 is but one of the many pattern sets made by the children.

Task B Talk about the rods in each row in as many ways as possible (several variations were discussed). See Figure 2.

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