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Towards Recording

Nick James and John Mason

Behind the formal symbols of mathematics there lies a wealth of experience which provides meaning for those symbols. Attempts to rush students into symbols impoverishes the background experience and leads to trouble later. In conjunction with manipulating objects it is essential to provide time for talking about their activities and developing their own informal records before meeting the formal symbols of adult mathematicians. We present three examples of children's work which demonstrate these steps in the struggle to move towards recording perceived patterns.

To most people the formal symbols used in mathematics seem cold and lifeless. Even the ubiquitous x is literally an unknown quantity with little meaning. Mathematicians often seem content to lend credence to this view by talking about mathematics as a formal game. This view of mathematical symbols is misleading in its incompleteness. In fact, for symbols of any kind to be of value there must be a wealth of background experience which can be called upon. This article is concerned with developing that rich background experience in the important phase of mathematical learning/investigation which we call TOWARDS RECORDING. The struggle to capture an insight which is as yet pre-articulate is often overlooked in a rush to lead students into formal symbols, resulting in an impoverished if not empty background experience, and producing frustration, anxiety, and math-phobia. We present three examples of childrens' work when time was taken to let the children participate in the struggle towards recording. The results reveal something of the stages in that process.

Keith and Ranjit (age 12½)

Task A Choose a rod and make up the equivalent length using repeating patterns (an example was given). Figure 1 is but one of the many pattern sets made by the children.

Task B Talk about the rods in each row in as many ways as possible (several variations were discussed). See Figure 2.

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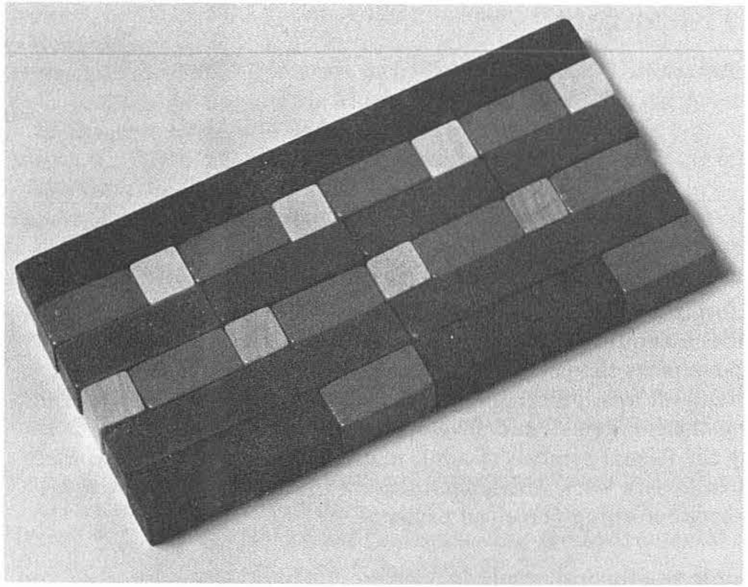

Figure 1.
Color of rods down left side:

purple rod
pink rod
light blue rod
white rod
dark blue rod
red rod

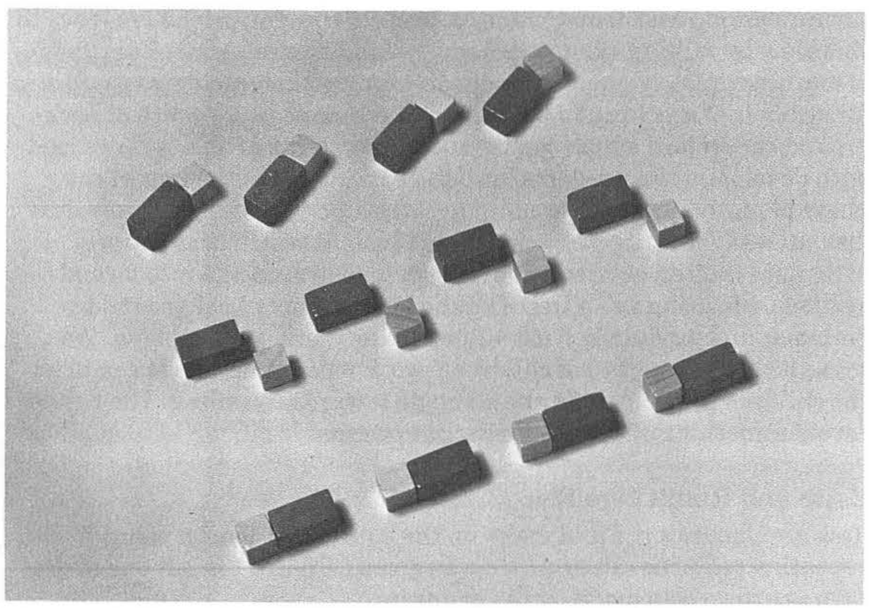

Figure 2. Discussion led to the second row from top being described as
"Pink-and-white . . . (pause) . . .four times"
"Four pink and four white"
"Four. . . (pause) . . . white-and-pink"

1ST ROW KEITH AND RANJIT
= 4 pink and White = Purple Rod
= 4 pink and 4 white = " "
= 4 pink + white x 4 = " "
= Pink + White x 4 = " "
= Pink, white, pink, white, pink, white, pink white
= 4 white and Pink
= 1 pink plus a white x 4
= 4 pink + white joined together
= Pink + white x 4

Figure 3.

Task C Write down the various ways of talking about the rods. See Figure 3. The teacher then asked them questions such as, "If I read Pink + White × 4, what rods would I put out?" All agreed on one pink followed by four whites. The attention was drawn to the need for agreement and precision. The crucial task came in a subsequent session.

Task D What must you write to leave no doubt about which rods you mean the reader to put out. Explore this amongst yourselves. Figure 4 shows what Ranjit wrote.

Purple Rod =
4 Pink and 4 white joined together x~~4~~
= 1 Pink and 1 white joined together x 4
Pink, white x 4 = 4 Pink and 4 White joined together

Figure 4.

Sensing that Keith and Ranjit had struggled to express an as yet pre-articulate sense of brackets, the teacher then offered Figure 5.

1 pink and white joined x 4

Figure 5.

Figure 6.

Pink and white × 4

= 4 Pink and 4 White joined together

Figure 7.

4(Pink + white)

Further discussion elaborated this to the idea of four bags each containing a pink and a white, and drawings like Figure 6 were adopted into their records. Later the balloon or bag was partly erased to yield the more common bracket (Figure 7). Furthermore, the children had begun to get a grasp of the distributive law (Figure 8). Further experience of this activity and also others involving the distributive law helped Keith and Ranjit to establish a strong sense of the use of brackets and the meaning of the distributive law. This is what is meant by a wealth of experience supporting the distributive law. The same process emerges in the next two examples which are based on investigations presented to two different age groups.

Lesley (age 9½) in a group of six children

Task A With 17 interlocking cubes make a square picture frame. All the while the children talked among themselves: "Two pillars of five down the side and a band of three along the top and bottom." After many trials they finally decided that it was impossible to use all 17 cubes to make a square picture frame. The teacher then asked them about other frames and after much discussion, other examples were produced (Figure 9).

Task B How many cubes are needed to make larger frames? Record your results! Figure 10 shows Lesley's results. Notice that the compelling nature of the underlying pattern has deflected Lesley from answering the original question. This will emerge if Lesley is invited

Figure 8.

Pink, white × 4 = 4 Pink and 4 White joined together

or 4 (Pink + white) = 4 × Pink + 4 × white

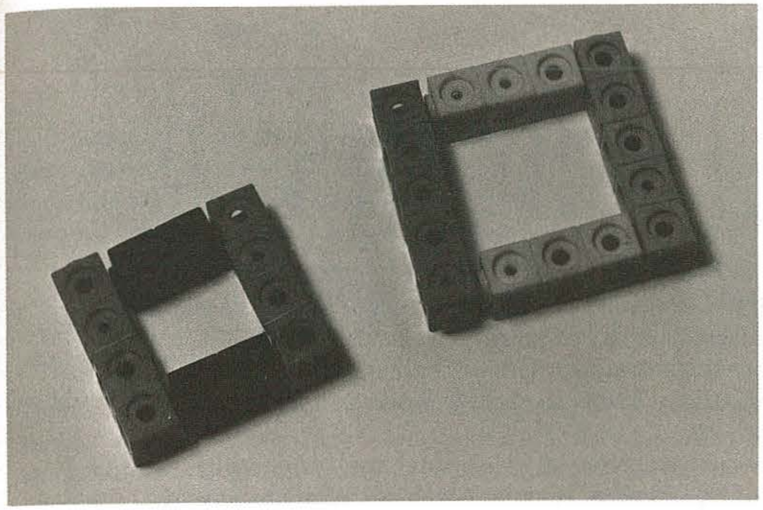

Figure 9.

Figure 10.

1 & 3 1st
2 & 4 2nd
3 & 5 3rd
4 & 6 4th
5 & 7 5th

Teacher intervention

"How many cubes for the 10th frame?"

6 & 8 6th
7 & 9 7th
8 & 10 8th
9 & 11 9th
10 & 12 10th
11 & 13 11th
12 & 14 12th
13 & 15 13th

Teacher intervention

"That's fine, but now, without carrying on, how many cubes for the 20th and 40th frames?"

After much thought Lesley made this conjecture:

"It's 20 and .... well, like 13 for the 13th and 15, which is two more. So it's 20 and 22."

She checked by carrying on and then wrote:

20 & 22 20th
40 & 42 40th

Lesley Barrett

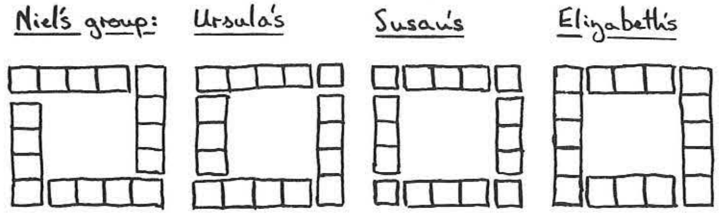

Figure 11.

to check her resolution against the task set. Time spent recording results is well rewarded though. In this case Lesley has come to her generalization from the support of her neat and systematic record. Unfortunately most adults are very reluctant to record results and consequently they find it harder than necessary to come to generalizations.

Patterns such as picture frames which invite generalizations lie at the heart of pre-algebra, because algebraic symbols arise in response to a need to record a generalization succinctly. The next task sequence shows how this can happen.

From picture frames to algebraic expressions

Task A Explore the range of possible picture frame designs to surround a 3×3 photograph. The children got together in groups and spent fifteen minutes producing a series of rough sketches drawn on squared paper. Much discussion of the possibilities ensued. Some of the sketches produced are shown in Figure 11. Each group in turn then described their designs to the rest of the class. Before attempting to generalize their designs to cover frames for square photographs of any size they need further experience of some other special cases. The teacher then suggested:

Task B Decide on a particular design and produce a whole range of picture frames, including the next smaller and the next larger ones. Figure 12 shows what Susan's group produced.

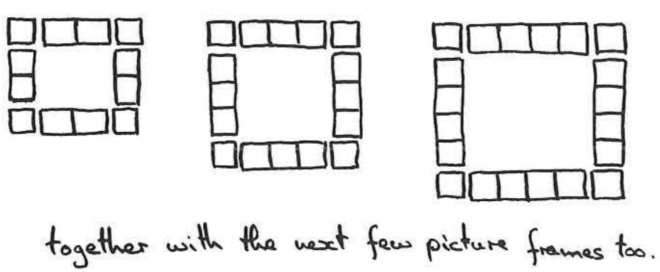

Figure 12.

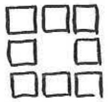
Figure 13.

Discussing their range of frames, Susan's group eventually added the smallest frame of all (Figure 13) saying: "This must be part of our collection. It's got four squares in the corners just like all the others. It doesn't have long bars in between though. They're only one long, but it's built the same as all the rest." They even paused to discuss the case of one four square at each corner with bars of length zero in between! This was finally rejected because such a frame couldn't contain a picture.

Task C Can you say how many cubes you'd need to make a frame to surround a square picture of any given size? Ursula's group offered this attempt at a generalization: "You must leave one square in the corner. On the left you have a rod the same size as the side of the picture in the centre. Then for the other three you just add one onto the size. You've got three of those." Once again the pattern Ursula's group was using to make each of their picture frames deflected them from answering the question in Task C.

Task D Write down how you'd find the number of cubes needed to make a frame for a square picture of any given size. Figure 14 shows what Ursula's group wrote. The teacher asked what they meant by "size of the picture" and they explained that it was whatever size the person happened to be thinking of. "If it was a 100×100 square photo then what would it be?" the teacher asked. The group explained: "One, for the corner; plus 100, for the size; plus $(101) \times 3$ for size + 1 taken 3 times; that's 404."

The children's explanation suggested to the teacher the idea of a "thinks cloud," like those used in the comics, to represent "the size of the picture I'm thinking of." He offered the cloud idea to the whole class as a means of refining the rather wordy generalizations they were all producing. He drew a stick person with a "thinks bubble" coming from its head to illustrate his point.

1 in the corner
Size of picture
Add one onto size and take it 3 times
Then all these added together.

Figure 14.

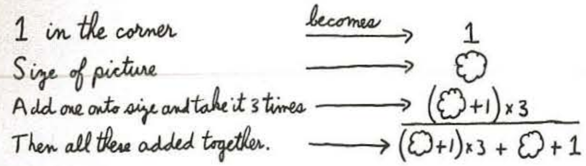

Figure 15.

Task E Try and shorten your statements using the notion of a "thinks cloud." Ursula's group added arrows to what they had already written pointing to the cloud notation (Figure 15). Having negotiated this shorthand with the class, it was a relatively short step for the teacher to introduce the more usual algebraic notation. "Whilst everyone in this class knows what we mean by 'cloud,'" he said, "the world's mathematicians use a letter like 'n' to stand for 'the number we're thinking of.'" Ursula's cloud shorthand was readily turned into Figure 16. The same process of recording and refining was carried out for the other groups (Figures 17 and 18).

the number of cubes needed to surround a square picture of side n is:

$$3(n+1) + n + 1$$

Figure 16.

The purpose of these examples has been to indicate the delicacy of the period leading up to adoption of a standard symbol system. Building on children's experience of doing specific tasks with apparatus, diagrams, or previously mastered symbols and depending on attempts to articulate to each other and to the teacher what they are doing, the act of moving towards recording is what gives substance to the otherwise heartless symbols of mathematics. Crucial aspects in order for the struggle towards recording to be meaningful include: (i) Minimal teacher intervention except to plant the seeds of helpful language patterns and recording devices, (ii) Sufficient time for the children to get to grips with the task, (iii) A minimum of three distinct instances from which a generalization can be formed, (iv) Encouragement of children to do tasks and talk about what they are doing, (v) A neutral environment permitting children to make conjectures which may be modified, without the stigma of being right or wrong.

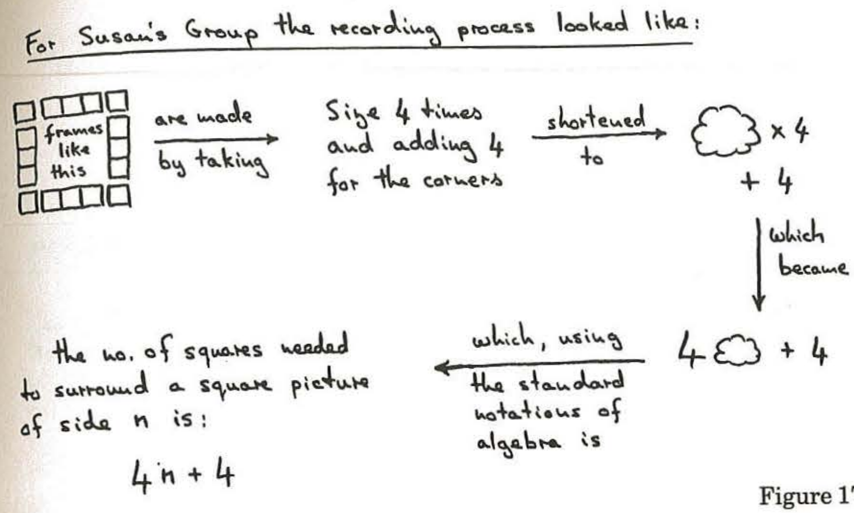

Figure 17.

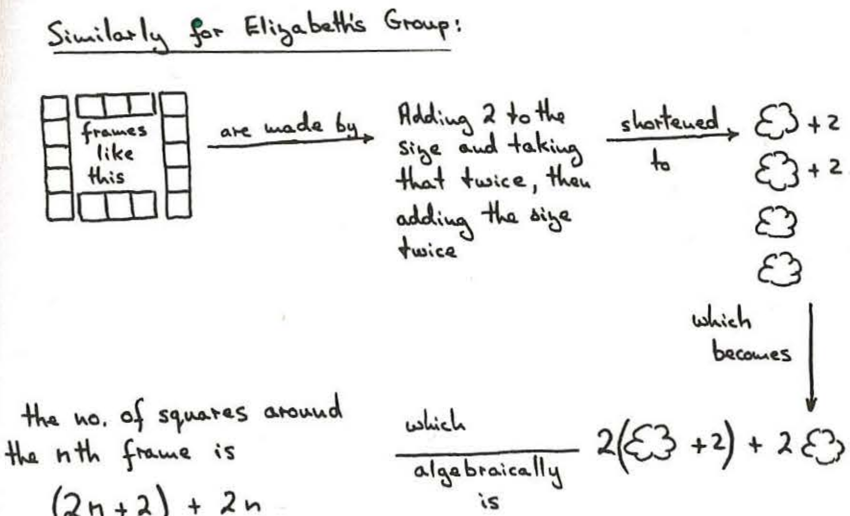

Figure 18.

The activities of doing, talking, and recording are classroom activities which facilitate the corresponding shifts in psychological states described in Mason (1980), moving from

Enactive to Iconic, that is from confident manipulation of specific instances to getting a sense of a common generalization;

Iconic to Symbolic, that is articulating the sense of generalization as a sequence of conjectures which are modified until they crystallize into an articulate and recorded statement which captures the notion.

The transition from Symbolic to Enactive, that is from an abstract form which is constantly referred back to examples to recall its intention, to a confidently manipulable entity which can serve as a component in a new, higher order notion,

requires practice to achieve mastery of the symbols. This is the true role of exercises in the mathematics classroom.

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Mental Images and Arithmetical Symbols

L. Clark Lay

Experiments by psychologists have led to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. An analysis is begun of the mental images that are judged to be properly evoked by certain number symbols of arithmetic. A variety of graphical models are suggested for use in linking these symbols to the desired mental construct. Some of these models have been found to be advantageous and may prove to be critically essential in certain mathematical contexts. Their assets and liabilities are discussed, and suggestions are made for modifications of conventional curriculum practice. A rich field of investigation exists in the visual imagery that can be associated with elementary mathematics. Progress here holds promise of extending mathematical competence to a larger portion of society.

The role of imagery in human thought has been studied by Piaget and Inhelder (1971), particularly as it relates to Piaget's well known genetic model of intellectual development. Their experiments led these authors to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. In our paper an analysis is begun of the mental images that are judged to be properly evoked by certain symbols of arithmetic. The emphasis will be on graphical models that can be used to link such symbols to the desired mental construct.

An experiment

The reader is invited to join in the following experiment. Writing materials such as a pencil and paper should be available. In a moment you will be presented with a very familiar symbol. You are asked to respond to this symbol, in the following manner:

Imagine yourself giving a verbal explanation of the meaning of this symbol to a person for whom it is not as yet familiar. Assume that the verbal discussion has not gone as well as you had hoped, and that it has occurred to you that a sketch or diagram of some sort might be helpful. You are asked to show your choice for this purpose. It is of particular interest that you record the first image that comes to your mind when this symbol is presented. If, upon further reflection, you

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