The activities of doing, talking, and recording are classroom activities which facilitate the corresponding shifts in psychological states described in Mason (1980), moving from

Enactive to Iconic, that is from confident manipulation of specific instances to getting a sense of a common generalization;

Iconic to Symbolic, that is articulating the sense of generalization as a sequence of conjectures which are modified until they crystallize into an articulate and recorded statement which captures the notion.

The transition from Symbolic to Enactive, that is from an abstract form which is constantly referred back to examples to recall its intention, to a confidently manipulable entity which can serve as a component in a new, higher order notion,

requires practice to achieve mastery of the symbols. This is the true role of exercises in the mathematics classroom.

Reference

Mason, J. H. (1980). "When is a symbol symbolic?" *The Learning of Mathematics, 1* (2 Nov.), 8-12.

Mental Images and Arithmetical Symbols

L. Clark Lay

Experiments by psychologists have led to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. An analysis is begun of the mental images that are judged to be properly evoked by certain number symbols of arithmetic. A variety of graphical models are suggested for use in linking these symbols to the desired mental construct. Some of these models have been found to be advantageous and may prove to be critically essential in certain mathematical contexts. Their assets and liabilities are discussed, and suggestions are made for modifications of conventional curriculum practice. A rich field of investigation exists in the visual imagery that can be associated with elementary mathematics. Progress here holds promise of extending mathematical competence to a larger portion of society.

The role of imagery in human thought has been studied by Piaget and Inhelder (1971), particularly as it relates to Piaget's well known genetic model of intellectual development. Their experiments led these authors to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. In our paper an analysis is begun of the mental images that are judged to be properly evoked by certain symbols of arithmetic. The emphasis will be on graphical models that can be used to link such symbols to the desired mental construct.

An experiment

The reader is invited to join in the following experiment. Writing materials such as a pencil and paper should be available. In a moment you will be presented with a very familiar symbol. You are asked to respond to this symbol, in the following manner:

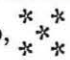
Imagine yourself giving a verbal explanation of the meaning of this symbol to a person for whom it is not as yet familiar. Assume that the verbal discussion has not gone as well as you had hoped, and that it has occurred to you that a sketch or diagram of some sort might be helpful. You are asked to show your choice for this purpose. It is of particular interest that you record the first image that comes to your mind when this symbol is presented. If, upon further reflection, you

Visible Language, XVI 3 (Summer 1982), pp. 259-274.
Author's address: 160 Calumet Avenue, Aurora, IL 60506.
0022-2224/82/007-0259$02.00/0

can think of other sketches you might use, we will be interested both in their variety and in the order of their coming to your mind.

Ready? The symbol to which you are to respond is "5"; the numeral for the number five. What image did 5 first evoke in your mind?

As an alternate experiment, the word five can be given orally, although I have not found this to affect the results to a significant extent. For the past many years the author has tried this experiment with subjects of wide diversity of attainment in mathematics, ranging from primary school pupils to university graduate students. When the study is limited to the initial response, there has been a uniform consistency in the type of diagrams that are drawn.

With very few exceptions the image that seems first to come to mind is that of an array of five separate but similar objects. These may be just five vertical lines, | | | | |, or these may be tied together as the tally , or there may be an arrangement in a characteristic pattern such as for a domino,  . Other subjects may show the fingers and thumb of one hand, or they may represent a collection of recognizable objects such as flowers, apples, or rabbits.

It would seem that even for those who have acquired a considerable sophistication in mathematics the symbol 5 is first perceived in its relation to counting as enumeration. But there is a considerable variety of ways to think of five. Some of these are not only advantageous but may even be critically essential in certain mathematical contexts. And these situations need be no more complex than those commonly introduced in the elementary schools. A list of twelve such representations of the number five appears at the close of this paper. These will be discussed in turn.

Numbers as counters

The first five letters of the English alphabet can be listed as; a,b,c,d,e. The acceptance of this collection of letters as a single whole can be aided by enclosing the given list by braces, { }. A temporary name, such as the letter S, can then be assigned to this collection, or set, of letters. Let #( ) be an operator; a symbol which directs that the number of members in the set be determined by counting. We then say that the number five is thus represented as the cardinal number of a set.

$S = \{a, b, c, d, e\}; \#(S) = 5$

During the mathematics education reforms of the 1960's this set representation of numbers was widely advocated, even for the

first introduction to number concepts. For various mathematical investigations, particularly those at an advanced level involving infinite sets, the advantages of set language and symbolism had already become widely known and accepted. It was hoped that the use of these mental models might also be an enlightening experience for the young learner as well.

But trials failed to support this innovation. Indeed, after a time, "sets" became almost a synonym for "what's wrong with the new math?" In retrospect it can be seen that set representations were tools that were too delicate for the tasks assigned to them; there were too many niceties to be observed in their use; so that confusion was often increased rather than decreased.

As an example of the care that must be taken, note that one needs to differentiate between a list and a set. Thus the list *a,a,a,* is different from the list *a,a,* and from the single listing, *a*. But, going back to the set S above, the notation used is defined as a roster notation: that appearing between the braces is a listing of *names*. But names used in this manner must be distinguishable; a repetition of the same name would introduce ambiguity. Hence $\{a,a,a\}$, $\{a,a\}$, and $\{a\}$ must then all be accepted as representing the same set.

To return to number as represented by an array of counters, we can anticipate trouble with the number zero. For centuries people must have thought: If there are no objects to be counted, what is the need of a number for this situation? Menninger (1969) found no trace of a written symbol for zero earlier than a Brahmi inscription of AD 870, although he states that the Sanskrit language had a name for this idea in *sunya* (empty) in the sixth century, and that the astronomer Ptolemy (about AD 150) used an abbreviation of a Greek word to indicate a missing place when writing fractions of Babylonian origin. Dantzig (1941) conjectured that zero was first conceived by an ancient scribe who wished to record an empty column on his counting board. Menninger puts it this way: "The zero is something that must be there to say that nothing is there."

If, as suggested, most persons associate numbers very strongly with the counting of objects, it is understandable why zero is often known only as nothing (no-thing). The set representation of numbers introduces the empty set as a model for the number zero. But again, it is just too easy for the beginner to confuse the emptiness (no-things) of this set with the set itself; since the empty set is only a convenient mental fiction, but nevertheless must be considered to be something (some-thing).

Counting of changes

The number zero has a much improved status when *changes* rather than objects are counted. Events can be considered as changes of state. Zero is then the number assigned to the original or initial state; before any of the changes to be counted have taken place. For such counting, zero is no longer tied to an absence of things, but rather to a lack of change.

In Figure 1 (adapted from Lay 1977) contrast is shown between the counting of objects (above the line), and the counting of changes (below the line). With the latter, five is now represented by a counting sequence. The arcs between the numerals for this counting sequence are meant to suggest changes of any kind that take place. A number is not assigned to the change while it is happening; the count is recorded only after the change is complete.

Counting objects	0	1	2	3	4	5
		*	*	*	*	*
Counting changes of state	0 ⁀	1 ⁀	2 ⁀	3 ⁀	4 ⁀	5

Figure 1.

There is a wealth of familiar activities and experiences which provide reason for counting changes of state. A simple example would be the counting of changes of position, as by steps. Zero then designates the starting position, the number 1 is recorded after the first step, 2 after the second, and so on. These further observations can be made for the comparison between the counting of objects (above the line) and the counting of changes (below the line).

zero	none, no object
	initial state, origin
one	object
	change, transformation
+1, unit increase	join one object
	advance to the next state
−1, unit decrease	remove one object
	return to the previous state

The concept of a counting sequence was used by Dedekind (1888) and Peano (1889) in their developments of a logical foundation for the principles of arithmetic. Such sequences are based on very fundamental intuitions. The questions to be answered are: "What comes first?" and then repeatedly, "What comes next?" A child is beginning to grasp these ideas when he or she can repeat, "Mary had a little lamb." Figure 2 suggests some of the significant reorientation that must take place when the imagery for numbers is to be shifted from counting to measuring.

Figure 2. From counting to measuring

* * * * *	
1 2 3 4 5	0 1 2 3 4 5
Counters	Scale
Counting	Measuring
How Many?	How Much?
Multitude	Magnitude
Separate	Connected
Discrete	Continuous
Natural Numbers	Non-negative Real Numbers

A small proportion of the persons who have participated in the thought experiment for numbers, as previously discussed, have sketched a scale for the number 5, similar to that in Figure 2. But most have not thought of doing this, even when encouraged to do so by leading questions.

One disturbing fact has come out of verbal discussions of such simple scales. There are persons who believe that Figure 3 is really a model for number six, rather than for five!

Figure 3.

Apparently they are so committed to the counting of objects that they react by counting the scale division points, rather than counting the line segments, or in thinking of the measure of the length of the entire segment. School authorities recognize the widespread avoidance by pupils of all the physical sciences, because of the reputed difficulty of these subjects. Much of the data for these sciences comes from measurements of quantities which the mind conceives as being continuous; such as mass, time, and measurements in space. What is the barrier to success for beginning students in these sciences? Can it be partially attributed to the pupils' lack of appropriate mental images for the symbols they encounter?

Models for rational numbers

The first extension of the number system, beyond that for the counting numbers, has traditionally been to the non-negative rationals, commonly known as fractions. Let us repeat the symbol response test, this time for the fraction, two-thirds.

What type of sketch or diagram first comes to your mind as being useful to communicate the meaning of $2/3$?

It can be anticipated that nearly all people will first draw a unit of some kind; it's "oneness" being suggested by its appearance of being "all there." Examples might be a circle, or a pie, or possibly a square figure. This is then divided in three parts of equal size, and the attention is directed to two of these subdivisions, by some device such as shading. For a verbal description we may say that two-thirds has thus been shown as representing two of the three equal parts of one (one unit, or all of something). But the fraction $2/3$ also represents one of the three equal parts of two, although a figure to illustrate this interpretation is very rarely given by subjects for our experiment. If the two is imagined as referring to two separate objects, this figure has a forbidding aspect if one is contemplating dividing it into three equal parts. This should be compared with the ease of *thinking* about a length (with a measure of two units), and sub-dividing this into 3 parts of the same length.

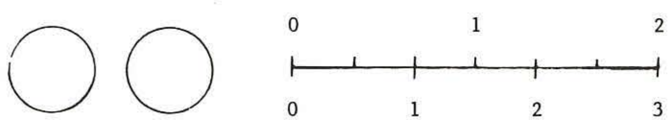


Figure 4.

The key strategy here is to take advantage of the arbitrary length that can be assigned to the measure of one unit. We begin with a line segment with designated points that are equally spaced. The zero and 1 of the scale are then located so that this assumed unit length can readily be subdivided into the prerequisite number of parts. With this done, then any positive integral multiple of this chosen unit length can be easily divided into the same number of integral parts. Thus in Figure 4 the number 1 was located to show the unit length divided in three parts; this assured that the length with measure 2 could also be so divided. There is a striking difference in the conceptual difficulty of thinking about dividing a two foot length of string into three equal lengths, as compared to thinking about dividing two apples into three equal portions. If this example with $2/3$ does not seem sufficiently impressive, one need only try contrasting the discrete and continuous models using slightly larger numbers, such as $3/7$. An important pedagogical advantage can be noted for the models using scales. A variety of illustrative examples are easily constructed by pupils, who are the ones who need the practice. But for the models for which the units are separated, both text and teachers are limited to the simplest of cases.

When the symbol x/y is interpreted as x of the y equal parts of one, this concept is commonly termed the parts-of-a-whole meaning. When x/y is thought of as the measure of one of the y equal parts of x units, we are appealing to the quotient meaning for x/y. Of these two interpretations, the quotient meaning appears much more frequently in applications but seems to be far less familiar to most adults. My hypothesis is that this handicap is strongly associated with the lack of the mental imagery that visualizes numbers as measures, such as their use on a linear scale.

The symbol x/y has still another interpretation, and its application extends to an even broader field than the two meanings already mentioned: The symbol x/y is also used to represent the ratio comparison of x to y. In part A of the Figure 5 we have a model for thinking about how 2 compares to 3. If a difference comparison is used (by subtraction), we say that 2 is 1 less than 3. But with a ratio comparison (by division), we say that 2 is $2/3$ of 3.

This same ratio comparison of 2 to 3, or of $2/3$, is also shown by diagrams B and C. Of the three, diagram C is considerably more flexible in its application. This flexibility arises from this distinctive property of ratio comparison: The ratio comparison of two magnitudes is independent of the scale used to measure them. Not only is C of Figure 5 a representation for the ratio meaning of $2/3$; it serves equally well for $2{,}000/3{,}000$ and for $.02/.03$, as well as $1/1\frac{1}{2}$ and $1.6/2.4$. An older notation for the ratio of 2 to 3 was 2:3, but there is increasing use of the same form as for fractions and quotients.

Scales for the measurement of length need not be confined to straight line segments. They are also used with curved figures, in particular with arcs of circles. Many phenomena in life are cyclic in nature; the same succession of events is repeated over and over again.

Figure 5.

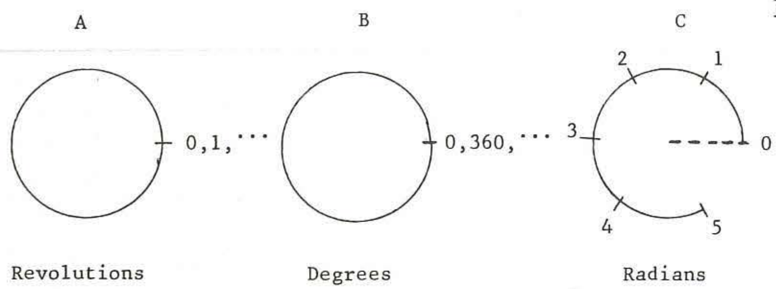


Figure 6.

This is strongly suggestive of traveling around and around in a circular path.

For a rapidly turning wheel or axle, it is convenient to assign the number one to a single complete turn or revolution. Let one revolution be divided into 360 equal parts, or degrees, as the ancient civilizations have taught us. Then many fractional parts of a turn are now measured with whole numbers: ½ turn is 180 degrees, ⅓ turn is 120 degrees, ¼ turn is 90 degrees, etc.

Another way to assign measure to circular arcs is shown in C of Figure 6. In a certain sense we permit the circle to decide its own measure. The size of a circle is fully determined by the choice of the length of its radius. Imagine a flexible tape on which the distance from zero to one is the same as the length of the radius of the circle, that is, the distance from the center of the circle to the circle itself. Begin at some point zero and wrap the tape around the circle. Then as in C of Figure 6 we have a picture of 5 as given in radian measure. This mental image of numbers is invaluable for many applications of mathematics in the field of calculus. This positive number 5 is measured in a counterclockwise direction; negative numbers are measured clockwise.

The association of numbers with ratios is very ancient, going back at least as far as the Greeks. Sir Isaac Newton (1769) considered the idea of ratio to be so basic that he used it in the definition of number: "By number we understand, not so much a multitude of unities, as the abstracted ratio of any quantity to another of the same kind, which we take for unity."

This way of thinking about numbers was given a concrete model by the Belgian educator, G. Cuisenaire, who introduced the colored rods which now bear his name. In Figure 7 if the white rod is assumed to have a measure of one, then 5 will be the measure of the yellow rod. The rods are unmarked, being identified only by color. This encourages a wide generalization: If any rod is assigned any positive number,

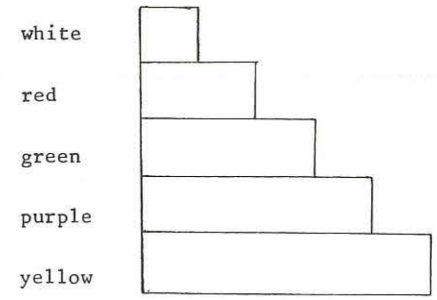


Figure 7.

then the ratio relations fix the unique number to assign to each of the other rods. For example, the purple rod is always twice as long as the red. If a number is assigned to either of these rods, then a number is fixed to assign to the other. Because of the three interpretations that can be given to x/y, these rods can be used to exemplify many properties of fractions and quotients, as well as ratios. Space does not allow discussion of their limitations, although the lack of a zero rod is evident.

Some non-linear models

The Cuisenaire rods vary in only one dimension of space, that of length. For two dimensions, with width or height as well as length, the square of unit sides and unit area is the fundamental unit. This is a difficult step for the learner in mathematics. Just a glance at Figure 8 is enough to reveal serious shocks to our intuitions. Certainly it is hardly evident that a square of area 4 is exactly twice the size of a square of area 2. Nor is it apparent that an area of 2 square units combined with an area of 3 square units should be equivalent to an area of 5 square units. A considerable amount and variety of "hands on" experience is a prerequisite before such relations can be made reasonable to our minds. Again, the difficulties with the ratios of areas are intensified when the ratios of volumes are considered. Here our intuitions are so strained that some might want to question the accuracy of the drawings for Figure 9.

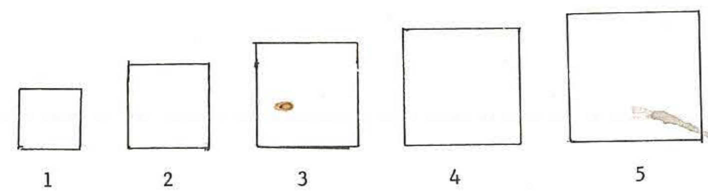


Figure 8.

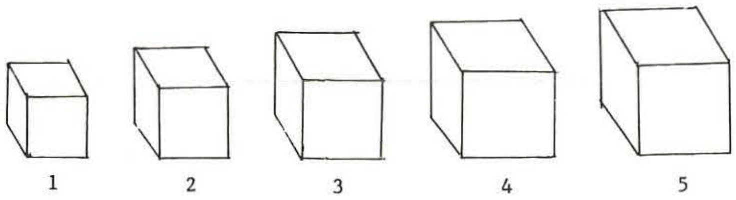


Figure 9.

Numbers as points

The symbols we call fractions—either the common variety such as $\frac{3}{4}$ or those called decimal fractions such as .75—are the numerals for the positive rational numbers. These numbers present conceptual difficulties far greater than those that arise when only counting numbers are considered. One of these mind stretchers is the loss of "nextness," which was an essential feature of the counting sequence in Figure 1.

What is the next larger fraction after $\frac{2}{3}$? The answer is this: Such a number does not exist; it cannot even be imagined. True, $\frac{3}{4}$ is slightly larger, but $\frac{17}{24}$ is larger than $\frac{2}{3}$ or $\frac{16}{24}$, yet $\frac{17}{24}$ is also smaller than $\frac{3}{4}$, or $\frac{18}{24}$. In fact, if x and y are two unequal rational numbers there is an infinite list of numbers that lie between them. For such a reason the system of rational numbers is said to be *dense.*

How can the mind be expected to visualize a dense set of numbers? The best answer we have is to adopt still another way of thinking about numbers, as suggested in Figure 10. On a line, extended in either direction as necessary, two distinct points are chosen. The number zero is assigned to one of these points, and the number one is assigned to the other. The line segment whose endpoints are zero and one then becomes the unit of length. The methods of geometry allow us to locate other points by adding, subtracting, multiplying, and dividing duplicates of this unit segment. Points determined in this way are rational points on the number line; each is associated with a unique rational number. The number five is now a point on this line.

Such a mental construct provides a model for a dense set of numbers, such as all the rational numbers between 1 and 4; we think of them as points on the line segment whose endpoints are 1 and 4. The points of a line do form a dense set. Between any two distinct points on a line there is another point, and even an infinite number of points. We have said that with each rational number there can be associated a unique

Figure 10.

point. But this does not mean that to every point can be assigned a rational number. Since Greek times it has been known that there is a number between 1 and 2, called a square root of 2 and written as $\sqrt{2}$, which is needed for measurement but that cannot be a rational number. Such numbers, and the points matched to them, are said to be irrational. Rational points are dense everywhere along the line, yet modern research has concluded that the set of points missed by the rationals—the irrationals—is also dense and in some sense there are even more of them than there are of the rationals. Thus we see how our search for a mental image for numbers has led to some very deep and imponderable properties. When we grant that for each counting number there is always a larger number, we are led to infinity in the large. When we reflect on the number line, we are confronted with infinity in the small.

For the counting sequence in Figure 1 every number has a unique successor. And every number, *except zero,* has a unique predecessor. If we allow every number to have a unique predecessor, the result is the sequence of integers. The notation for the integers is symmetrical, with zero as the center of symmetry. With each counting number different from zero there is paired its opposite; that is named by the same numeral, but by also including a minus sign as a tag to distinguish it from its partner.

The counting numbers provide answers for "How many?," and the rational numbers do this for "How much?" The extension to the negative numbers is useful for "Where?"

In Figure 11 the sequence of integers is used to extend the numbering of points for Figure 10; thus completing a figure that is called the real number line.

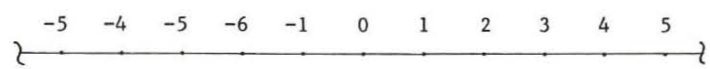


Figure 11.

Vectors for Numbers and Operators

There is still another visual image that we can use with great advantage for our example, the number 5. For each point on the real number line, except zero, we can associate a directed line segment, or vector. This vector will have its initial point at zero and its terminal point at the number by which it is named. The arrowhead at the terminal point gives the vector a sense of direction which is lacking for an undirected line segment. If the point zero is accepted as a limiting or degenerate vector, the vector model for each real number is then complete.

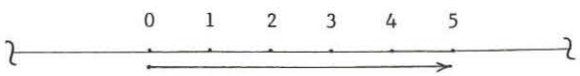


Figure 12. The vector should lie along the line, but is offset here for clarity.

Before presenting our final representation of number, we mention a caution to be learned from Skemp (1971) in his instructive discussion of mathematical symbolism and imagery. A visual symbol can convey a distinctive and even a dramatic message, but it can also be imprecise in this communication. Two persons looking at a mathematical symbolism or a visual model of it do not necessarily "see" the same thing. A simple and yet very fundamental example can be given.

There are several ways to think about even such a simple form as 3 + 2. We have first been taught to think of this in terms of a binary operation (an operation on two numbers); which can be emphasized by underlining, 3 + 2. Three is a first number and two is a second. The plus sign links these to form the composite symbol, 3 + 2. The arithmetic student may be encouraged to think of the plus sign as suggesting an operation (addition) yet to be done, to get the number 5 which will be called the sum of 3 and 2. Yet the algebra student must accept this addition as already accomplished by the writing of 3 + 2. This is a result of accepting 3 + 2 = 5 as a true statement because 3 + 2 and 5 are names for the *same* number. The change from 3 + 2 to 5 must be recognized as a change of form but not a change in amount. It is unfortunate that elementary texts commonly gloss over this conflict of meanings.

But the end is not yet for 3 + 2. This time we underline to suggest a unary operation meaning, 3 + 2. Three is still a first number, but the composite symbol + 2 now represents not a number, but rather a change. We think of + 2 as an operator, representing an increase of 2. The number 3 is the operand. When the operator + 2 is joined, by writing it at the right, the result is the transform, 3 + 2, which represents a second number.

Unary operations have possibilities for introducing a dynamic point of view into arithmetic which is yet to be recognized by texts and teachers. Curiously enough, some of the present practices already seem to follow the unary operation concept. For example, in presenting 3 + 2 = 5, a textbook picture may show a static model for the

number 3, such as three children standing in a group. But many such illustrations include a dynamic model, not for 2 but for + 2, as shown by two children at a distance but running to join the others. This leaves the number 3 + 2, or 5, to be imagined as the state after the two groups have become one.

Even the vertical display for suggested addition computations has a slight bias toward a unary operation.

3
+2 , suggests a 3 + 2, a unary operation

3
+2, would be a better binary sumbolism

If we return now to Figure 12, a fortunate circumstance can be observed. The vector model for the positive number, 5, can serve equally well as a mental image for + 5, that is, for an increase of 5. For a decrease of 5, as indicated by the operator − 5, the vector would have the same length as for + 5, but with the arrowhead moved to the other end. In summary, the length of the vector can give the amount of the change, while the two senses of direction along the line can differentiate between the two opposite kinds of change.

However, the single vector shown in Figure 12 is too limited in its portrayal of an increase of 5. The vector shown there is also the position vector for the number 5. As such, it is a bound vector, with its initial point necessarily at the origin.

But we want to think of increases as beginning at any chosen number (or point on the line). For this we need a free vector, that is, a vector free to move along a line but without changing its length or sense. We therefore enlarge our vector concept to include an equivalence class of vectors. (Two vectors are equivalent if they agree in length, direction, and sense.) As long as these conditions are met the vector remains equivalent even though it is translated to a new position.

The vectors for Figure 13 all represent increases of 5, even though they have been shifted to the left or to the right. Again we are to think of their acting along the line, even though they are here moved down for clarity.

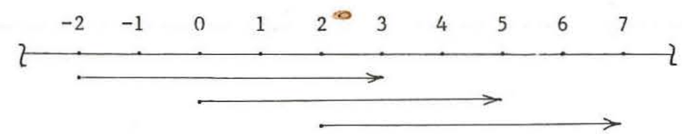


Figure 13.

As given here, our final interpretation for the real numbers will be to identify each with an equivalence class of vectors. For our example, the number 5, these will be the vectors of length 5 that are directed in a positive sense (as from zero to one). Space does not allow a demonstration of how this correspondence between numbers and vectors provides a foundation for the study of signed numbers.

Also omitted is the necessarily extensive discussion of the various binary and unary operations on numbers. We would find that there are distinctive advantages and disadvantages for each of these varied visual models as we consider such operations as addition, subtraction, multiplication, and division. Our purpose has been limited to suggestion of the rich field of investigation that exists in the visual imagery that can be associated with numbers.

Our society presents an ever increasing demand that mathematical competence be extended to a larger portion of its members. To make this possible we need to seek a better understanding (a better mental picture?) of number and its uses, and this properly begins with a study of its simplest ideas.

References

Dantzig, T. (1941). *Number, The Language of Science.* Macmillan, New York.

Dedekind, J. W. R. (1888). *Was sind und was sollen die Zahlen?* Brunswick.

Lay, L. C. (1977). "Much ado about nothing," *Illinois Mathematics Teacher, 2,* no. 3, 12-13.

Menninger, K. (1969). *Number Words and Number Symbols.* MIT Press, Cambridge.

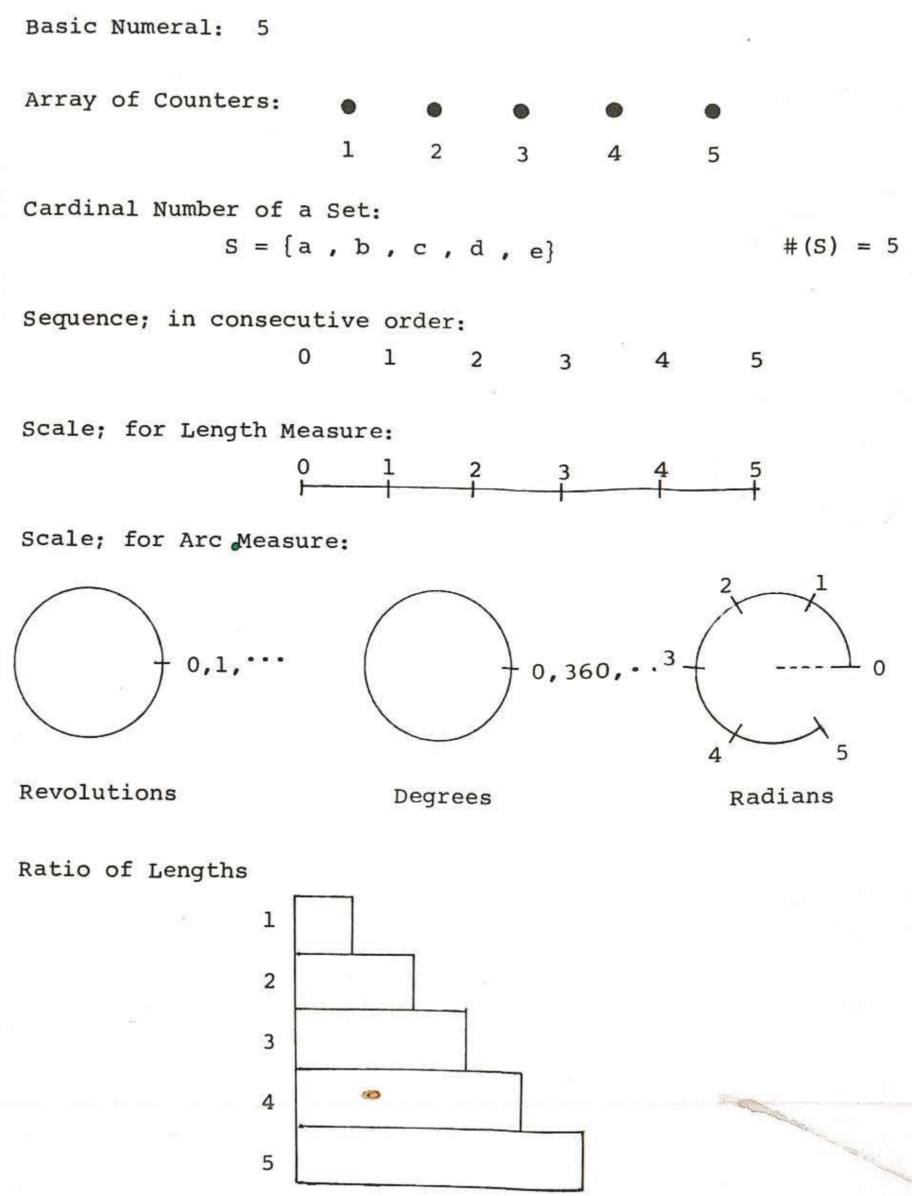
Newton, I. (1769). *Universal Arithmetic.* W. Johnson, London.

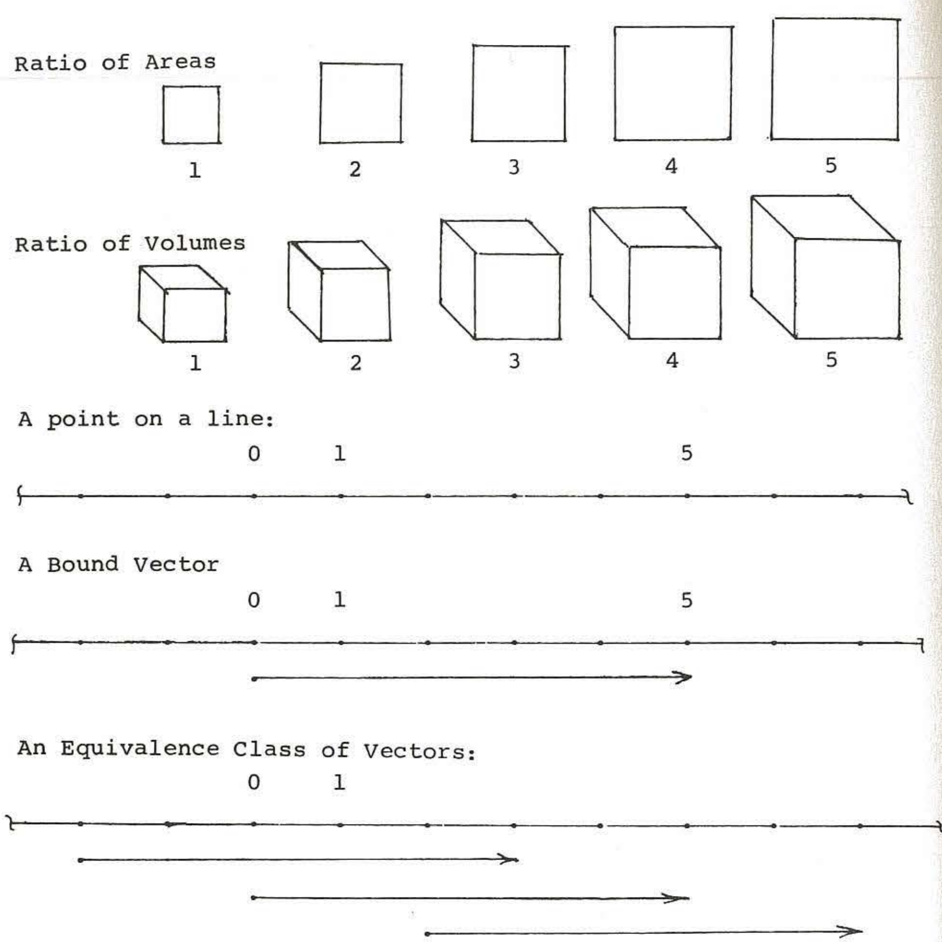
Peano, G. (1889). *Arithmetices principia, nova methodo exposita.* Turin.

Piaget, J., and Inhelder, B. (1971). *Mental Imagery in the Child.* Basic Books, New York.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics.* Penguin, Harmondsworth.

Representation of number





An equivalence class of vectors can also correspond to an increase of n, as represented by the operator + n.

Language Acquisition through Mathematical Symbolism

Francis Lowenthal

We noticed that the use of a non-verbal formalism can favour cognitive development (in the frame of the elementary school) in problem children as well as in normal children. An example is given to show how a formalism inspired by mathematics can be used to aid the development of the verbal language of 8- to 9-year-olds. We will then analyze the results and try to discover the cause of success we observed.

First we must specify which symbolic systems and which mathematical formalisms to use. In a previous paper (1980a) we stated, "We think that the main factor of cognitive development is *manipulation of representations*." In another paper (1980b) we claimed that any representation system which satisfies the six following criteria can be used: the system must be *non-ambiguous, simple and easy to handle, non-verbal* (to avoid conflicts with the developing verbal language); it must also be *supple enough* to enable the child to become conscious of what he knows but cannot verbally express; it seems essential that such a system should be *suggestive of a logic* and could be introduced and used *in the frame of games* (to enable us to use it easily with young children).

We wanted each of our systems to be suggestive of a logic; this is why we decided to choose representation systems used in mathematics. This requirement enabled us to represent our symbolic system in terms of a game. The rules of a game are explained and the children must collectively build a representation. This is the first stage of their work: *the synthesis*. They must then modify the representation and only respect technical constraints while doing so. They then reach the last stage: the analysis of the new representation and the collective discovery of the rules of the new game. Similar exercises can be invented for language acquisition.

What follows is a report of an actual lesson during which we asked the children "to tell a coherent story corresponding to a given representation." We will thus describe the adventures of a class of normal 8- to 9-year olds. The representation system we chose is that which is used in the new math (Papy 1968). Objects are represented by

Visible Language, XVI 3 (Summer 1982), pp. 275-280.
Author's address: 24 Rue des Dominicains, B-7000 Mons, Belgium.
0022-2224/82/007-0275$02.00/0