VISIBLE LANGUAGE

The quarterly concerned with all that
is involved in our being literate

Volume XVI Number 3 Summer 1982

SPECIAL ISSUE

Understanding the Symbolism of Mathematics

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Special Issue

UNDERSTANDING THE SYMBOLISM OF MATHEMATICS

Guest Editor: Richard R. Skemp

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For those who understand and enjoy mathematics its symbolism is a gateway to an elegant, satisfying, and powerful mental apparatus. But for those to whom mathematics is a source of difficulty and confusion, these same symbols are more often perceived as barriers to understanding. Those who understand mathematics — who can attach correct mathematical meanings to its symbols — pay little attention to the symbols themselves as they pass beyond them to the associated mathematical ideas. But those who do not understand mathematics do not get beyond its symbols, which rightly or wrongly they regard as one of their main sources of difficulty.

My personal view is that though the power of mathematics resides in its ideas, access to this power is largely dependent on its notation, and that the better the notation the more effectively we can handle the ideas. (Compare the difficulty of multiplying CLXIV by XVIII with the relative ease of multiplying 164 by 18). Even for competent mathematicians, therefore, there is much to recommend the study of notation in its own right; and particularly, what are the properties of, and criteria for, a good notation. And for those concerned with mathematical education, a study of the particular problems of learners with respect to its symbolism would seem to be indispensable if help is to be given in an area where it is particularly needed.

The present collection of papers is offered as a contribution in this area, together with the hope that others too may begin to perceive mathematical symbolism as a subject likely to reward greater study than it has yet received. *R.R.S.*

Difficulties with Mathematical Symbolism: Synonymy and Homonymy

Josette Adda

We know that the confusion between meaning and sign (in French: *signifié/signifiant*) is the root of a great number of mistakes in mathematics. Particularly, instead of making easier the approach to the mathematical concept represented, the sight of the design often produces a disturbance to understanding; it leads to mistaking the *drawing* for the presented *idea,* as idolatrous people do. I will demonstrate — by presenting many genuine examples which I have met in mathematical classrooms at every level — the mathematical roles of synonymy and homonymy.

Mathematical objects are, by nature, abstract objects. Only through their denotations is it possible to encounter them. So, the problem of the linguistic relation of meaning (i.e., the relation between *signified* and *signifier*) is particularly crucial for mathematical understanding. Teaching and learning situations bring to light difficulties inherent to mathematics. Failures by students are signs of epistemological obstacles. So we are going to study our problem through paradigmatic cases, observed during mathematics classes.

Thinking of the role of symbols, we would be happy to have a one-to-one correspondence: SYMBOL↔MATHEMATICAL OBJECT. We would like: (1) that each symbol should denote one mathematical object and only one, and this not only for the teacher (or writer of a textbook or an examination) but also for each of the students — and the same one for everybody! (2) that each mathematical object be represented by one single symbol. Alas! It does not work like this (see, for instance, Skemp 1971, Freudenthal 1973, and Adda 1975-1976); so we will see that we cannot escape the linguistic problems of synonymy and homonymy.

I SYNONYMY

Synonymy of symbols is decisively related to the problem of identity. One never needs to say that one object is itself, but one often says that two objects are only one (for instance: "the *two* numbers a and b are equal so that they are the *same* number"). This is a frequent use, but it is a misuse because what we intend to mean is that the two *names* are names of the same single object, that they are synonymous.

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The symbol "=" is of constant use in mathematics and its function is to mean that the symbols written on its left and its right denote the same object. Thus 2 = 2,00 = 4/2 and this means that "2" and "2,00" and "4/2" are symbols referring to the same object, the same number. [Editor's note: As readers will know, the decimal point is rendered in French usage by a comma.] But many studies (e.g., Kangomba 1980) show that pupils, and even some teachers, often say that 2 is a natural number but neither a decimal nor a rational number, while 2,00 is decimal but neither natural nor rational and 4/2 is rational but neither decimal nor natural![1] If we refuse, as some people do, the complete identification by embedding the set of natural numbers into a part of the set of rational numbers we have to renounce definitely to write "4/2 = 2" which is very useful!

Being unaware of the synonymy, pupils can write without qualms: "2 + 3 = 5 + 7 = 12 × 2 = 24". This comes from a general use in school of questioning statements such as "2 + 3 = . . .", in which the symbol "=" does not have the same meaning as "equals" but rather that of "gives as result," and so the above statement can be understood as a sequence of manipulations on a calculator. Writing in the same way as when operating with a calculator is a sensible behavior but, unfortunately, it does not lead to correct mathematical statements.

Brookes (1980) notices that whatever they have been taught about this, when asked: "Look at '7 + 8 = 14'; correct the mistake, please" almost everybody has the same initial reaction and puts "15" at the place of "14"; far less spontaneous are other corrections such as putting "7 + 7" in place of "7 + 8", or even "8 + 8 = 16" . . . . This shows that the asymmetric meaning of "=" is pregnant for all of us. Furthermore, while the mathematician has emphasized that 2 and 2,0 are the same object in some later physics lesson the pupil will be told of a crucial distinction between 2 and 2,0 (about accuracy).

Many teachers and textbooks authors are disturbing. F. Cerquetti (1981) quotes a strange mathematics textbook in which, in an exercise, "2,10" is described as "incorrect writing":

8. Supprime les zéros inutiles:

écriture correcte	*écriture incorrecte*
2,1	2,10
. . . .	04,05
. . . .	30,100
. . . .	108,20
. . . .	0,00050
. . . .	1,2800
. . . .	104,0

Though, three pages down, fortunately, one can see the expression "2,50f" in another exercise!

II HOMONYMY

When two different objects have the same (or nearly the same) designation, problems of understanding are bound to arise, and we know of cases in which designations differ in spelling only and in which children, listening to a text which does not make sense for them, mispell it. It seems that children who have the greatest difficulties with the linguistic problem of spelling are also those who are unaware of its function; it would be fruitful to enable them to become aware of the importance of the convention.

As an example, an eleven-year-old French boy noted for his very poor language performances (especially in spelling) was, on the contrary, very bright when working with LOGO (Papert's computer display turtle). He decided to draw on the screen a camera, the program of which he called FOTO. But the drawing appearing on the screen was not satisfying, so he prepared a new program and called it FAUTAU, and after this a new one called FAUTEAU, and another, the good one, called FAUTTEAU. Most surprising is that this boy never did confuse the names when he was typing on the fingerboard. The spelling conventions decided by himself were very coercive for him. In mathematical language we often use the same (or quite similar) notations for different concepts and this creates difficulties unless the difference is suitably emphasized.

1 Confusion between similar notations

I shall precise this type of confusion by presenting examples about the symbolism of a sequence of figures followed by a comma and of a sequence of figures.

Emmanuelle (an autistic girl, ten years old, studying in a special class for mentally handicapped children) looks at the three exercises written on the blackboard by the teacher. In each of them two or three decimal numbers occur, some of them being written with commas (remember that this is the French use). In her exercise-book, she writes operations where not only the numbers appear to be combined as at random, but also the decimal notation is often decomposed and treated as a symbol denoting a couple of numbers.[2]

Well, will you say, this is a very extreme case! But what about the comparison between 5,2 and 5,18? Many experiments (e.g., inquiries by the IREMS of Rouen and Strasbourg) show that even fifteen-year-old pupils claim that $5{,}2 < 5{,}18$. Is not it provoked by the same confusion as Emmanuelle's? "5,2" is not seen here as another name for the number also written "5,20". Instead there is on one side the "5" and on the other the "2" which means less than "18"! In the opposite direction, we can find situations in which a couple of natural numbers is confused with a decimal one. For instance, I saw a 13-year-old pupil faced with the definition of integers as classes of couples of natural numbers* so that he had to perform additions of couples of naturals as an introduction to additions of integers. I observed on his paper the following mistake:

$$\begin{array}{r} (4,7) \\ +\ (3,5) \\ \hline =(8,2) \end{array} \quad \text{instead of} \quad \begin{array}{r} (4,7) \\ +\ (3,5) \\ \hline =(7,12) \end{array}$$

This can be compared with the following line that G. Glaeser saw in an examination of complex numbers (at first year of university):

$$\frac{1+i}{1-i} = \frac{(1,0)+(0,1)}{(1,0)-(0,1)} = \frac{1,1}{0,9} = 1,22$$

So even with the plain problem of figures with comma—with or without parentheses (a very small difference?)—we can see those confusions at many various levels of studies.

Another type of example of confusion between similar notations involves the case of notation by *nothing,* i.e., juxtaposition. Many pupils are troubled by its use for products, and so, being ignorant of the rules for algebraic writing, they are led to the following "simplification" well known by all teachers:

$$\frac{\cancel{6}}{3a\cancel{6}} = \frac{1}{3a}$$

2 Confusion between different linguistic levels

In section 1 the confusion was only by pupils on their own: they identified expressions which were not exactly identical. But now we will consider confusions in which teachers share the responsibility because of language misuses.

*For those unfamiliar with this definition of integerns, we recommend that this example be omitted. Eds.

Logicians emphasize the need to use the symbolism of quotation marks to distinguish the symbol, taken as object, from its referent. Actually, in writing, quotation marks are often omitted, and for oral discourse it is quite difficult to make them perceptible. One often misuses *referent* when one intends to speak about the *symbol*: For instance, consider the sentence, "An even number ends with 0,2,4,6, or 8." Here is the same confusion of linguistic levels as in the sentence: "Paris has five letters." Indeed, nobody is disturbed by the confusion between Paris which is a city and "Paris" which is the name of a city (here is a mine for jokes and puns from the most ordinary kind to Lacan's), nor by the confusion between the even number which is divisible by two and its symbol in decimal notation. But when we listen to a teacher in primary school, we are very surprised by the abuses made during the study of numeration in which both numbers and their denotations are considered. We often hear (and even read in textbooks) the following sentence: "To multiply a number by ten, add a zero." In this strange formulation, multiplication is an operation on numbers while addition is intended as an operation on sequences of figures (i.e., a metamathematical operation) and nobody tells the pupils that! So do not be surprised to hear some poor child saying: "2 + 1 are 21."

Confusions are frequent in the study of fractions. For instance, 1/2 and 2/4 are equal, but "1/2" has a prime denominator while "2/4" has not. If you say or write the previous sentence without quotation marks (as it is generally done), that will be quite disturbing for the meaning of equality.

In these examples, we have seen two levels confused in the same discourse. Even when there is only a single level, we can find misunderstanding in communication between teacher and pupil if one of them thinks at one linguistic level and the other one at another. The date with "1979" being written on the blackboard, I observed a teacher asking a nine-year-old (in a special class for the mentally handicapped) to write a larger number, the pupil wrote in the middle of the blackboard a very large: 352

In some sense he was right, but I had to convince the teacher!

Jaulin-Mannoni (1975) asked a child in front of this drawing

to draw "three times a" and he drew

and not three drawings of a tree as she expected. That certainly does not show a difficulty about multiplication but rather about the confusion between symbol and reference.

III THE PROBLEM OF VARIABLES

The situation of homonymy — in which a symbol is considered as meaning, at the same time, both its referent and the symbol itself — often occurs, as in the last example, with the use of letters for symbols of variables. For non-mathematicians this use is particularly disturbing. Actually, we are dealing here with the main specificity of mathematical language and, for people who failed in mathematical learning, the language was often the barrier where they got stopped. Baruk (1977) asked Daniel what an equation is; he answered, "It is figures and letters." We can hear others saying, "Oh mathematics! Some a, some b, and x, and ... equals zero." This is the only mark left by ten years of mathematics in much of the population.

This use of letters is an important difficulty inherent to mathematics. We cannot avoid it, but perhaps we can make it more explicit to pupils. We are simultaneously confronted with phenomema of homonymy and synonymy: apparent homonymy between the symbol and the signified (but we will see later on that some perverse exercises are based on it) and hidden synonymy between the letter and other designations of an object.

For instance, in "$2x + 3 = 0$", "x" is synonymous of "$-3/2$"; in (1) "$ax^2 + bx + c = 0$", the symbol "x" is synonymous of the two in "$-b \pm \sqrt{b^2 - 4ac}/2a$", "$c$" is synonymous of "$-ax^2 - bx$", and "$ax^2 + bx + c$" is synonymous of "0".[3] But all the letters have not here the same function: for instance, in an equation some letters represent *unknowns* and others represent *parameters,* depending on the role in the problems. Example: (1) is a second degree equation in x, or a first degree equation in c.

Furthermore, the meaning of a mathematical sentence depends on the structure which interprets the symbols, and each letter denotes any object of the reference set attributed to this letter. For instance, consider the expression:

"$\exists\, x\ (x + 3 = 1)$"

if "x" ranges over $\mathbb{N}$ (the set of natural numbers) the sentence is false, and if "x" ranges over $\mathbb{Z}$ (the set of positive and negative whole numbers) it is true, while

$\forall x\ (x + 3 = 1)$
(1)

is false over $\mathbb{N}$ and over $\mathbb{Z}$ and

$\forall x\, R(x + 3, 1)$
(2)

is true over $\mathbb{N}$ if "R" means the ordinary relation of order (so that the structure $<\mathbb{N}, \geqslant>$ is said to be a "model" of the sentence [2]), and not true if "R" means the relation of equality, etc.....

Davidov and Wilenkin (see Freudenthal 1974) conducted experiments in teaching the use of variables at the very beginning of mathematical studies, in elementary school. Pupils then seem to be more able to understand the meaning of operations; they are not distracted by computational difficulties when confronted with a word-problem with letters. It seems to be easier to teach them to substitute the writing of numbers to letters than to teach them, as we generally do, to generalize by letters the writing of numbers. For instance, when we say, "Let n be a number," we often hear children protest: "n is not a number, it is a letter." A good process is (as Varga does) to tell them "choose a number" (and each pupil chooses his own) and "do so and so ...." Then we explain what was done by each one by saying "the number," and quickly it becomes more comfortable to abbreviate and more natural to say "n" for "the number." Note that this process only works when they can forget that "n" is a letter.

So we have to stand up against the perverse exercises in which two linguistic levels are occurring: such as asking children this strange question: "What is the set of the x such that x is a vowel?" which mixes the letters considered as objects and the letters considered as symbols for variables. Do not be surprised if pupils are troubled and answer "$\emptyset$"! The convention to designate variables by letters was taken for mathematics about numbers not about letters (just to avoid designation of a number by another number!) Still worse are traps such as this classical one: "Calculate $(x - a)(x - b)(x - c)$ .... $(x - z)$," in which two levels are deliberately mixed. This could be a pleasant joke[4] but please be aware that answering "0" is not compatible with the mathematical use of variables. You have to think to the factor $(x - x)$ where, in a true mathematical situation, you would have to consider the difference between two numbers (independently of the alphabet of the country!).

Homonymy is also dangerously encountered in situations where we use the same letters for names of objects and for linked variables.[5] For instance, I observed a teacher referring to a linear map as "f" and following exercises with various special cases dealing with maps always called "f" as well, asking whether each f was linear or not. Pupils were completely disturbed, trying reasoning involving vicious circles (the property of the f being intended as given in the definition).

Points as variable symbols

The use of "etc. . . . " and of ". . . " can be connected with the use of variables. We encounter them with an infinity of references, and the authors hope that we shall discover the meaning through the context (see Adda 1979). Actually, questions such as "Complete 1,3,5,7,. . . " (though of frequent use by psychologists) are not mathematical because, mathematically, here ". . . " can mean anything. But pupils have to understand many expressions like:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

and the canonical interpretation of them can be difficult when pupils are not familiar with the context.

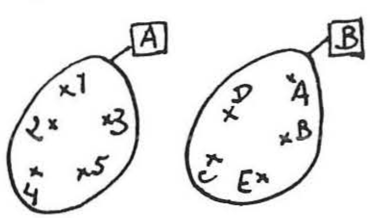
We find another use of points in "fill-the-blanks" exercises. For instance:

$$\begin{array}{r} 4\,. \\ +\ .\,5 \\ \hline =71 \end{array}$$

I think this is worse than letters because here the same symbol is used in cases where one would have written different letters.

BY WAY OF SYNTHESIS

To conclude, it might be interesting to examine the answer of a twelve-year-old boy to a teacher who had asked for an example of two disjoint sets:



Curiously, the teacher found it good! For me, this is a mathematical monster. Not only could the sets have a non-empty intersection and even be equal in many cases (for instance $A = 1, B = 2, C = 3, D = 4, E = 5!$), but let us not forget that necessarily $A = A$ and, even more, $B = B$! Is it not that the reason this child did not see this as a special monster is because for him, as for many people in the non-mathematical world, mathematical symbolism is considered as a sorcerer's code for which ordinary people cannot hope to discover a key?[6]

1. It seems to me that a large part of the responsibility lies in textbooks and in such teachers' expressions as: "A decimal number is a number *written* as . . . " and "A rational number is a number *written* as . . . " instead of "a decimal [resp. a rational] number is a number which *can be written* as . . . ."
2. Sometimes figures were recombined in other numbers so that, one day, I observed:

On the blackboard	On Emmanuelle's paper
13,5 − 9,5	−135 59 1 164

3. Of course it is not simple. Here is the situation of synonymy generated by *descriptions*. It is very complex. In the beginning of our century logicians with Bertrand Russell thought very much about paradoxes such as: "Walter Scott is the author of *Waverley*" so that "Walter Scott" is synonymous of "the author of *Waverley*," and in nearly all situations you can write one for the other — but not in the descriptive sentence itself, obtaining "Walter Scott is Walter Scott" which is a very different sentence!
4. One French textbook (by IREM of Strasbourg) presents it as a "poisson d'Avril" but not all teachers are so honest!
5. Also, though I did not observe cases of misunderstanding, we can note the use of the same letter for names of special objects and for variables with, for instance, "π" (taken as for the special number and as for names of planes . . .) or "i" (with $i = \sqrt{-1}$ or names of integers, . . .) etc. . . .
6. I am grateful to C. Berdonneau for many corrections of the poor English of my first version of this paper.

All the quoted examples are French. Those about commas are probably avoided by the notation of the decimal point. I am not able to know if there exists a similar perturbation for English and American pupils.

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Emotional Responses to Symbolism

Laurie G. Buxton

Special difficulties often arise in reading mathematics because of the symbols and notation that are used. This is caused not only by the range of symbols and their density of meaning (interiority) but also by strong emotional responses raised by certain symbols or combinations. These feelings may reflect unpleasant memories of when the symbols were first encountered, but may even derive from an unease with the shape of some of them.

Much learning hinges upon the decoding of symbols, for it is mainly by means of written symbols that the knowledge the human race has accumulated is stored. Most of us learn satisfactorily to read our own language, though any of us can be confronted with passages of prose or poetry which we are able to translate from the written symbols to the spoken word, but cannot claim readily to comprehend. On the whole we remain comfortable when presented with a piece of our own written language whose symbols do not, with some reservations discussed below, occasion us disquiet. However, how are they regarded by someone who has not been able to learn to read? The range of unpleasant feelings is considerable. The mere sight of symbols of language will occasion fear, distaste, embarrassment, and shame. Anyone who has sought to teach an adult illiterate will confirm that this statement is not too strong. There is, in fact, a vicious circle whereby the emotional response to the symbols is such as to inhibit the individual's cognitive processes, which may in themselves be perfectly adequate to acquire the skill of reading. It is difficult to put oneself in the position of a non-reader (or even a pre-reader, though we have all passed through this stage). But once we introduce mathematical symbols, most of the population can be put precisely in this situation.

I shall describe three experiments on reactions to symbols, and hope then to offer explanations of two of them. The first was conducted with various groups of people most of whom knew some mathematics and had a generally positive attitude to the subject. The following statement was shown on a screen by an overhead projector:

$$\phi(x) \text{ is continuous for } x = \xi \text{ if, given } \delta, \exists$$
$$\varepsilon(\delta) \text{ s.t. } |\phi(x) - \phi(\xi)| < \delta \text{ if } 0 \leqslant |x - \xi| \leqslant \varepsilon(\delta)$$

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and the assembled company was asked to read it. "Reading" meant turning the written symbols into speech; it did not imply comprehension. No-one was able even to "bark the words," let alone penetrate the meaning, so they were in the position, relative to this passage, of a genuine non-reader.

The group were asked individually to record their *emotional* response on seeing the statement. Some said "mystified" or "double-dutch," but there were a number of replies in the area of "fear," "anxiety," or "apprehension." This type of reaction, in fact, prevents people even being willing to attempt understanding. We may assume that the symbolism of mathematics, despite its many advantages, can induce feelings inimical to learning the subject.

At this stage it is worth distinguishing between symbols and notation. By symbols I mean single characters, such as ξ, but by a notation I mean a grouping of such signs to convey a particular meaning. When we write (3,4) the symbols used are common ones, but the particular grouping of signs has a great deal of extra meaning (or to use a Skemp term, "interiority") not detectable by anyone who merely knows the separate signs. The effect of this is to render the apparently common place rather mysterious. This is one of the features of the language of mathematics that makes it inaccessible to so many.

Returning to our first example of mathematical writing, it may be that the use of Greek letters accounts for some of the negative reactions. The second series of experiments with groups of people illustrates this, though there are other factors at work as well. When offered the suggestion, "Plot the point (3,4)," most of the groups I was dealing with were happy enough, in that they understood the notation and the instruction was clear to them. Not all were sure which way one should measure the 3 and which the 4, but that was the only area of unease.

With the statement, "Consider the point (x,y)," there was a sense of uncertainty and insecurity in some, deriving partly from the formality of the language and partly from the familiar numerals being replaced by slightly mysterious letters. Yet the statement was still on the whole acceptable.

Finally the group was presented with "Let $P(\xi,\mathfrak{y})$ be such that. . . ." Quite apart from the unfinished nature of the statement and the increased formality of style, the impact of $P(\xi,\mathfrak{y})$ was such as to render extinct any hope that what followed might be understood. One person claimed that once such a statement was stated, "The shutters came down" as far as he was concerned. In part this derived from the notation of setting the letter P next to the known notation, and in part from the Greek letters, which not everyone could even *say*.

Enough has perhaps been said to establish that the written language of mathematics has not only a density of meaning that renders understanding slow in coming but that the mere presentation induces an unease that will not allow one to make a start on penetrating the meaning. Those feelings laid down as a result of earlier failures in dealing with mathematical symbolism inhibit an appropriate cognitive attempt even at reading them.

There is another separate response that is of interest. In this third experiment, and working again with various groups of people, another emotional response to symbols has manifested itself. The evidence given so far suggests that unfamiliarity with Greek letters may be an important influence in producing negative reactions. Without refuting this, experiments in attitudes to single letters indicate that some are more acceptable than others and that certain Greek letters (such as α and ρ) are found to be pleasantly formed and quite appealing—and that this is not true of all the letters in our own alphabet. In presenting this issue to a group I discussed the fact that some relatively unfamiliar letters, such as α, did not seem to create unease. I then asked that the question of familiarity or frequency be cast from their minds and that each person should decide individually which lower-case letter of the English alphabet they found most *strange*. With every group of people, q emerges as the easy winner. Even with groups of as many as thirty people not more than six letters were mentioned, with x, z, k, and j appearing, but in every case with far fewer votes than q. It is not easy to guess what this may mean. Perhaps it is simply the shape. Certainly among the Greek letters ξ is not as easy to accept as ρ. Why should we respond in this way, and what effect does it have on our being able to deal with mathematics? At this stage I have not even reached a hopeful speculation.

So we see that all purely cognitive approaches to the understanding of mathematical symbols and notation will be ineffective unless they recognise that an emotional dimension exists. *Acceptance* of a symbol or a notation is an emotional issue. It may come simply with usage and familiarity, but mere definition will never suffice. Even with signs to which we have become accustomed there may remain a flavour of distaste which makes us less competent in their use.

So far the case is stated. If we now accept that there is a problem we have two things to do. The first is to give a rationale for why we should feel as these experiments indicate that we do, and the second is to suggest possible ways of preventing "symbol-fear" from arising, or (more difficult) remedying it when it has occurred.

Skemp (1980) has indicated that emotions may signal danger and an attack upon oneself. May we interpret the situation of reading a string of symbols in the light of this belief? Is an attack being made? Certainly all ones early learning experience leads one to believe that demands are being made. When presented with symbols, including the language ones, the demands is that some response be made, such as reading it, understanding it, or working something out. The threat lies at the end of that process, because failure to satisfy the teacher's wishes can often have a negative emotional outcome! Even if in a present situation no demands are being made, one's belief is that they are, based on previous experience. It may be that the symbols of mathematics make more difficult and heavier demands than other symbols. They are perhaps more functional, operational, active than the letters in which our prose is written. This is illustrated by the successive lines in which an equation is solved. So routine can become the various transpositions which we make that the symbols seem to have a life of their own in arriving at an answer. Even so, an illiterate probably does get quite strong negative charges from the printed work, and as we have said, most people are illiterate when it comes to "reading" mathematics.

The answer is not to avoid mathematical symbols in a child's earlier experience. Rather one should capitalise on situations where the children feel a need for symbols. Several examples may illustrate this.

A group of children in one primary school were playing with some chime bars and at the end of one day had found a tune which happened to please them all. They wanted to play it the next day. No sophisticated derivation of musical notation arose, but the five chimes were labelled *A, B, C, D, & E* and they simply wrote down a string of letters. The beginnings of a notation, and perhaps the first step towards algebra? Certainly a code they all understood, and not discoverable from outside without further information.

A second case, again with primary children, arose in recording journeys. In going from, say, school to public library at every junction they used one of three symbols L, R, or A. "A" stood for "ahead," "L" for "turn left," and "R" for "turn right." They worked happily with them and managed to establish how a string of symbols was transformed on the reverse journey. (Interestingly, if we introduce "U" for "about turn," we have a group isomorphic to that formed by the powers of i).

A last example comes from my own work with a group of experienced primary teachers. We were engaged upon an investigation of the regions created when straight lines cut each other, all in distinct points. We had arrived at a four-line configuration and at first I had labelled

the spaces by a string of capital letters. This was readily accepted. Suddenly I wanted to convey the number of boundaries of a region, and whether it was open or closed, and without preparation labelled it as shown in Figure 1. The effect of one member was to say that she was totally lost. The *need* for a notation should have been raised and the group should have been asked for suggestions.

Regular sessions, then, at all stages in mathematical education, of experiencing the need for a symbol or a notation and the discussion of the notation suggested, would greatly ease the situation. In most people's experience each addition to our complicated system has simply been produced like a rabbit out of a hat.

There is a related but distinct reaction to symbols that we may explain differently and perhaps remedy in other ways. A page of mathematics can induce not so much a clear and remembered threat as a feeling of insecurity, sometimes at a level that can be described as panic. A model of this feeling developed by Skemp and myself is described by myself (Buxton 1981). Briefly, a failure to comprehend the symbols places the reader in one area where is lost, with no sense of goals or direction, and with no sense of how to act appropriately. Panic ensues.

In general a symbol represents a concept, whereas a notation involves a whole schema lying behind it. We saw this in the plotting of a point described in various ways earlier. If the schema is not known to

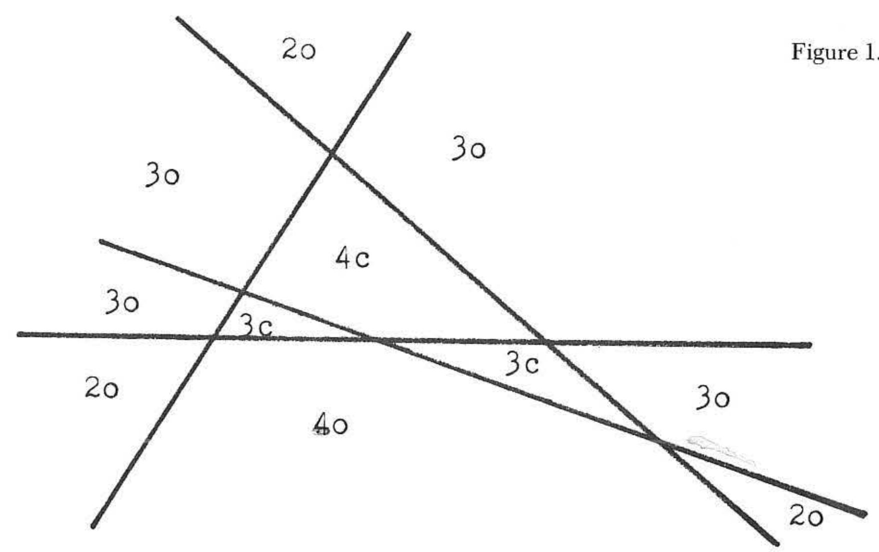

Figure 1.

the student, even if the separate concepts are, he will be unable to operate. Approaches to mathematics teaching that are largely content based will attempt to develop it logically and to develop all parts of the *subject* in an ordered fashion — and this is admirable. However, more important than the schema lying within the subject (in Popper's world 3) are those in the mind of the student (world 2) (Popper 1973). We need to check with the students whether they find that the information conveyed fits what they have in their minds. An interesting experiment is to ask a number of people whether "minus times a minus is a plus" fits comfortably into their minds. When the student believes, rightly or wrongly, that the idea does fit, then and only then should you move on. It is the "emotional acceptability" of what we are told or read that is the measure of whether we can advance.

Most teachers check out whether their students understand, and by this they are addressing the cognitive. It is necessary to ask whether they accept — and *that* is affective. Once the strings of symbols are attached comfortably to those patterns we already have in our minds, we are secure.

Finally we should mention one counter-indication to what we have said, and point again to one question discussed earlier but not resolved. In the discussion on (ξ,η) we did assume that the schema of two coordinate axes, and the plotting of points was known and comfortable. Why did the use of unfamiliar symbols induce discomfort? Perhaps it is felt that they must convey something more, something mysterious — else why were such letters used? But the reason is not clear.

As for why q is so "strange" — perhaps someone can help?

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Mathematical Language and Problem Solving

Gerald A. Goldin

Problem solving in mathematics may require different kinds of language: the verbal or mathematical language in which the problem itself is posed, the notational language of problem representations available to the solver, and planning language for heuristic reasoning and formulation of strategies. This paper explores some relationships among these languages, with examples of ways they can influence problem-solving processes.

I Introduction

Problem solving in mathematics refers to situations in which some items of information are given or available, and one or more goals are described. The problem solver is expected to attain the goal(s) through logical or mathematical procedures. Sometimes the term "problem solving" is restricted to the case in which the solver has no routine algorithm available for this purpose. Mathematics educators have become increasingly interested in studying problem solving and improving its teaching (Polya 1962 and 1965; Harvey & Romberg 1980; Krulik 1980; Lester 1980).

Kilpatrick (1978) proposed to organize the independent variables of problem-solving research into three main categories — subject variables, task variables, and situation variables — for the purpose of understanding how problem-solving outcomes depend on variables in each category. A collaborative study of task variables was conducted by a number of researchers (Goldin & McClintock 1979). In this work the characteristics of problem tasks were considered under the following headings: syntax variables, describing the grammar and syntax of the problem statement; content and context variables, describing the semantics of the problem statement; structure variables, describing mathematical aspects of a problem representation; and heuristic behavior variables, describing heuristic processes associated with or intrinsic to specific problems. Task variables were taken to be independent of the individual problem solver, and defined instead with respect to a population of solvers. They are subject in principle to control by the researcher or the teacher.

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Let us now distinguish among various kinds of language which can be employed during problem solving: the verbal language of the problem, notational languages, and planning language.

A Verbal Language

The verbal language in which the problem itself is posed may be a natural language such as ordinary English, and may include technical terms from mathematical English. Task syntax variables are descriptive of this language. Barnett (1979) reviewed a large number of studies on syntax variables, organizing them into the following categories: variables describing problem length; variables describing grammatical complexity; formats (verbal or symbolic) of numbers or other mathematical expressions; variables descriptive of the question sentence; and the sequence of information in the problem statement. Linear regression studies have indicated that variables of length and grammatical complexity, defined in various ways, do affect the difficulty of verbal problems in arithmetic (Loftus 1970; Beardslee & Jerman 1973), but have provided little insight into how this occurs.

The problem statement is often descriptive of a "real-life" situation which can be pictured or visualized. Content and context variables, reviewed by Webb (1979), describe the semantics of the problem statement. The term "content" refers to mathematical meanings, and the term "context" to nonmathematical meanings, insofar as this distinction can be maintained. Sometimes a problem posed in words may be accompanied by a picture or diagram; then we regard this picture as part of the problem content or context.

B Notational Languages

Notational languages available for problem solving, unlike ordinary language, are highly structured formal systems. They may have strict semantical rules for writing well-formed expressions, and a well-defined set of allowed transformations from one expression to another. Examples include the notations for our system of numeration, for arithmetic operations, for fractions, decimals and percents, for algebra, trigonometry and calculus, for set theory and symbolic logic, and diagrams picturing allowed constructions in Euclidean geometry. Evidently a great deal of the teaching of mathematics is devoted to communicating the rules for working within such languages.

Once a problem has been translated into a notational language, purely formal manipulation of symbols according to the rules of procedure is usually sufficient to arrive at a solution. Nevertheless, the symbol-

manipulation may continue to be motivated by visualization of the "real-life" situation that the notation now describes.

The concept of a problem state-space has been employed to describe the mathematical structure of problems, as well as to map the behavior paths of subjects (Goldin 1979). As defined by Nilsson (1971), a state-space for a problem is a set of distinguishable problem configurations, called states, together with permitted steps from one state to another, called moves. A particular state is designated as the initial state, and a set of goal states is distinguished by the conditions of the problem. When a problem can be translated into a standard notational language, the mathematical sentence or diagram which is the most direct translation becomes the initial state. A notational language thus provides a standard representational framework in which the state-spaces of many problems can be embedded. Sometimes for non-standard problems, the solver is in effect presented with a new notational language, with simply stated rules of procedure, and the object is to "learn" the language in proceeding from the initial symbol-configuration to the goal. A problem state-space is thus a notational language in miniature.

C Planning Language

Finally we have the language available to the problem solver for heuristic planning or formulation of strategies. This is the language in which the solver establishes subgoals, organizes trial-and-error search, seeks analogous problems, or engages in the many other forms of planning described by Polya. Thus it is a language *about* problem solving as well as a language *for* problem solving. It appears that children and adults engage in heuristic planning to a considerably greater extent than they can describe explicitly. One of the goals of protocol analysis in studying problem-solving behavior is to describe from an information processing standpoint the planning which occurs, based on a transcript of a subject's "thinking aloud" statements. It would be valuable to systematize such language so that it could be used in the teaching of problem solving.

Figure 1 shows the various levels of language available for problem solving. The perspective of this paper is to treat all of the levels of language as "existing" apart from the individual problem solver, defining them in relation to a population of problem solvers sharing a common "mathematical language."

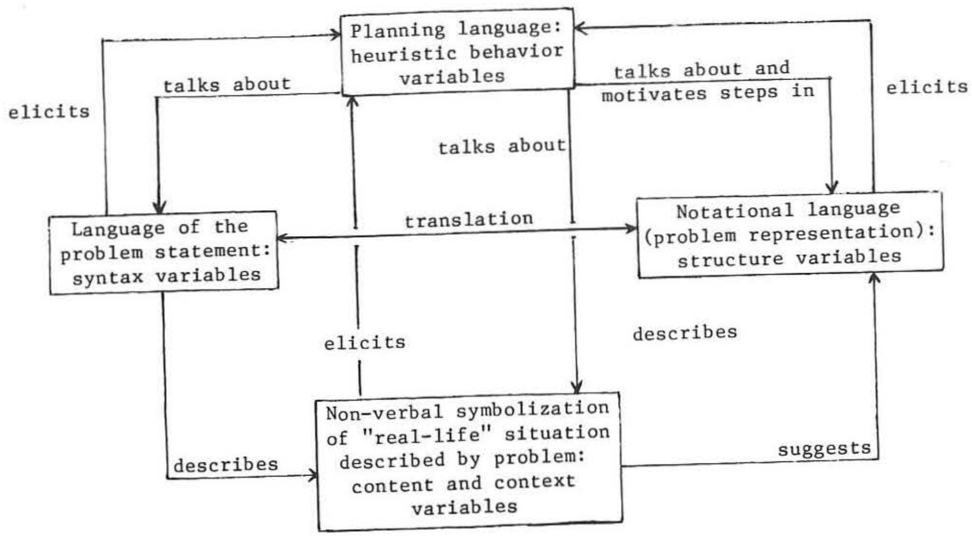


Figure 1. Relationships among levels of language available for problem solving.

II Examples

In this section we describe two examples which illustrate the concepts introduced above. In addition they illustrate the important point that *small* changes in the statement of a problem can result in *large* changes in problem-solving processes, even when the mathematical structure of the problem is held fixed.

A Plants and Flowerpots, Cats and Dogs

The following problems were used with elementary, junior high, and senior high school students (Caldwell & Goldin 1979; Goldin & Caldwell 1979):

1 Alan bought an equal number of plants and flowerpots. Each plant cost three dollars and each flowerpot cost five dollars, so that he spent 48 dollars in all. How many plants did Alan buy?

2 Jane has an equal number of dogs and cats. If she had twice as many dogs and four times as many cats, she would have 42 pets in all. How many dogs does Jane have?

The two problems were originally intended to be parallel, except that the first problem is stated factually and the second has a hypothetical component. It turned out that Problem *1* was less difficult than Problem *2* for every school population studied, but not necessarily for the reason expected.

The languages of the problem statements have values for syntax variables which are quite close. Both problems have three sentences; Problem *1* has 33 words and Problem *2* has 34 (excluding articles). Both problems contain three items of numerical information, with the first two (small) numbers written in words and the third (larger) number written as a numeral. The grammatical complexity of the two problems is comparable, as measured by a "syntactic complexity coefficient" developed by Botel, Dawkins, and Granowsky (1973). The question sentences occur at the end of both problems, and are of exactly parallel length and grammatical construction. The two problems differ in syntax in the factual/hypothetical variable. There are other minor syntactic differences as well; the second problem, for example, uses the pronoun "she" twice, while the first uses the pronoun "he" but once.

The notational language of algebra provides a standard representation for each of these problems (unlikely to be available, of course, to the students in elementary or lower junior high school grades). With the obvious choices of letters for unknowns, Problem *1* translates to: $P = F$, $3P + 5F = 48$; while Problem *2* translates to: $D = C$, $2D + 4C = 42$. These two systems of equations can be solved in exactly the same manner, and in exactly the same number of steps, to yield $P = 6$ (for the first problem) and $D = 7$ (for the second). We therefore say that in this representation, the two problems have the same structure. An alternate notational language, often used by younger children, involves the use of "guess and check" procedures. For example, the child may first make a "guess" as to the number of plants, and compute the total cost. If this is too low, a new "guess" is made. Schematically, we have something like this: "If 1 plant, 1 flowerpot, $3 + 5 = 8$, too low; if 2 plants, 2 flowerpots, $3 \times 2 = 6$, $5 \times 2 = 10$, $6 + 10 = 16$, still too low; . . . . " until the trial "6 plants" occurs. This procedure can also be used to find the number of dogs in the second problem. Some children are able to carry out these procedures aloud, without the use of written notation at all. Whether written or oral, it is convenient to think of the procedure as occurring in a formal language containing, for example, "trial" statements and "comparison" statements acting on a domain of whole numbers (the "search space"). Such procedures have been examined by Harik (1979).

The planning which takes place when these problems are solved is often silent. The algebra student may say, "First I will write down some equations, then I will solve them," and the grade school student may comment, "Let's try some numbers." Along the way, additional

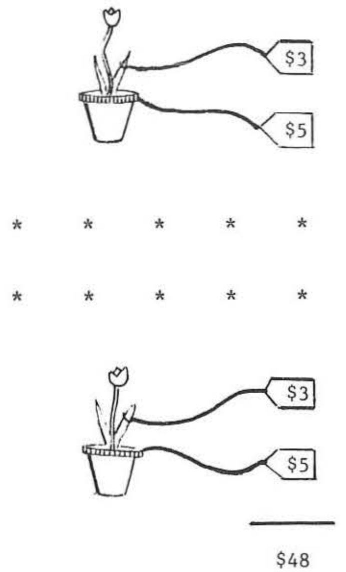


Figure 2a. One way to visualize the plants and flowerpots.

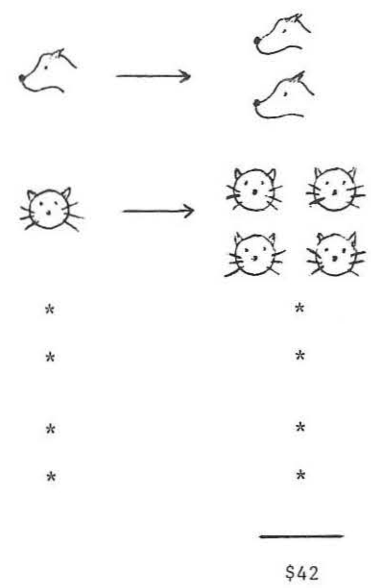


Figure 2b. One way to visualize the cats and dogs.

planning may occur aloud; for example, "Skip some numbers." Most often the observer is left to infer the nature of the planning which occurred, through analysis of the solver's verbal protocol.

Turning to the process of translation from the problem statements to algebraic notation, we note that these problems contain "key words"— words which very frequently translate to particular mathematical operations. For example, the phrases "Each . . . cost" and "times as many" translate to multiplication (×), while "in all" translates to addition (+). Since such terms occur nearly in parallel in the two problems, students who translate directly from the problem statement to notational language (as in Figure 1) should arrive at parallel systems of equations.

On the other hand, the real-life situations described by the two problem statements are quite different. Figure 2 depicts one way in which these may be visualized. This difference allows the following method of solution for Problem *1*, which is not available for Problem *2*. In Problem *1* the picture suggests: "Each plant cost $3 and each flowerpot cost $5, so that the pair cost $8. Since Alan spent $48, he bought 48 ÷ 8 = 6 plants." The analogous line of reasoning for Problem *2* is extremely awkward to phrase or to visualize, even though the problems are of corresponding mathematical structure. For this reason, the original intent of creating problems which were parallel except for the factual/hypothetical variable was not entirely achieved. Referring again to Figure 1, the language of the problem statement described a "real-life" situation which in turn suggested a notation (3 + 5 = 8, 48 ÷ 8 = 6) different from that obtained by direct translation, and in this case more efficient.

B A Checkerboard and Paper Clips

This well-known problem provides a second example for discussion:

> *3* Consider an 8 × 8 arrangement of squares, from which diagonally opposite corner squares have been removed (Figure 3). A paper clip may be placed so as to cover two squares adjacent horizontally or vertically, as in the illustration. Can all the squares be covered by paper clips without overlap? If so, how; if not, why not?

The problem statement describes a concrete apparatus which itself can serve as a notation for making moves. Often solvers proceed to experiment by placing paper clips, until after several trials they acknowledge their inability to achieve the goal. During this stage of problem solving, little overt planning may occur. Atwood, Masson, and Polson (1980) discuss a model for problems which are similar to this one in that successor states are generated from an initial state by application of a single rule of procedure. Their basic assumption is that subjects do not plan, but use only information from the current problem state and those which immediately follow to make each move. In a study of "water jug" problems, they found their model to account adequately for subjects' behavior. It may well be the fact that a notation is provided by the problem itself which encourages subjects, at least initially, to restrict themselves to mechanical moves within the notation.

In Problem *3*, however, planning is necessary if the solver is to proceed beyond the observation that the trials do not succeed. More

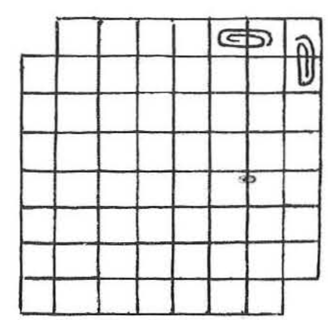


Figure 3.
Diagram for Problem *3*.

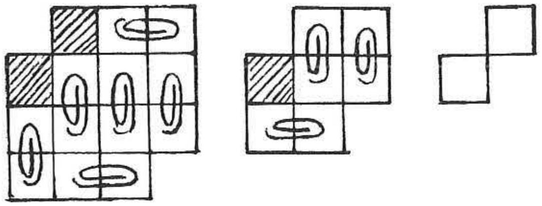


Figure 4.
Trying to solve
a simpler problem.

sophisticated or "educated" problem solvers might even engage in planning from the start. For example, the heuristic advice, "Try to solve a simpler related problem," may lead to examination of the 2×2 case (clearly impossible), the 3×3 case (impossible since there are an odd number of squares), and the 4×4 case, which is quite similar to the given problem but allows much more rapid exploration (Figure 4). Trials on the 4×4 case may lead to the observations that diagonally attached squares often remain uncovered after a trial, and that the same squares seem to remain in a variety of trials.

One way to achieve insight into this problem is to *improve* the notation by coloring those squares which remain after various trials. The decision to do this requires the ability to think or talk *about* the language being used to represent problem states; i.e., to think on the level of planning language. The pattern of colored squares which results is that of an ordinary checkerboard. Now it can be observed that a paper clip always covers a colored square and a white square. Since in the initial 8×8 problem there were 32 colored squares and only 30 white squares, and they are being reduced in equal numbers, the squares cannot all be covered by paper clips—there will always be two colored squares left over.

Possibly Problem *3* would be less difficult if its statement referred to "an 8×8 checkerboard" instead of "an 8×8 arrangement of squares," or if one set of squares were shaded in the diagram. The original notation was less effective because essential information was not *visually* apparent in the representation of a state (although it could have been obtained of course by counting). Again a small change in the problem statement, which does not affect the problem structure, suggests a substantial change of notation which in turn facilitates the problem solution.

III Efficient Notational Language and the Structure of Problem Representations

This section first looks at examples of efficient and inefficient notation in standard representational frameworks. Then we examine how, in

non-standard representations, the choice of symbolism can illuminate or conceal important structural features such as problem symmetry, or affect the complexity of each move.

A Standard Languages of Mathematics

Much of the progress of mathematics across history is attributable to the development of improved systems of numeration and modern algebraic notation. Arithmetic problems which would have posed formidable challenges in ancient Greece or Rome can be solved by today's school-children. The process of experimentation and notational change is an ongoing one today in algebra and analysis. From the perspective of problem solving, an effective notation should have certain characteristics, among which are the following: (1) Symbol-configurations should be reasonably concise, with information most likely to be important made visible rather than suppressed. The number of steps needed to move from one configuration of symbols to another should be small. (2) To the extent that concepts are parallel mathematically, they should be represented in parallel syntactically. Two examples will illustrate these points.

When the "new mathematics" was introduced in the 1950's and 1960's, precision of meaning in notation was sometimes emphasized at the expense of problem-solving effectiveness. The "raised minus sign" was introduced to denote negative numbers (additive inverses), and $^{-}3$ was called "negative three," not "minus three." "Minus" was reserved for the operation of subtraction, with "$8 - 6$" defined as "$8 + {}^{-}6$." Operations such as addition, subtraction, multiplication, and division were treated strictly as binary operations (acting on two numbers at a time), and each step had to be justified with reference to the appropriate structural property of the number system (associative property for addition, commutative property for multiplication, etc.). A consequence of rigid adherence to these rules might be the following sequence of steps in algebra: $3X + 7 = 19$ [given], $(3X + 7) + {}^{-}7 = 19 + {}^{-}7$ [addition of the same number to equals yields equals], $(3X + 7) + {}^{-}7 = 19 - 7$ [definition of subtraction], $(3X + 7) + {}^{-}7 = 12$ [renaming], $3X + (7 + {}^{-}7) = 12$ [associative property for addition], $3X + 0 = 12$ [additive inverse], $3X = 12$ [additive identity], $(1/3)(3X) = (1/3)12$ [multiplication of equals by the same number yields equals], $((1/3)3)X = (1/3)12$ [associative property for multiplication], $((1/3)3)X = 4$ [renaming], $1X = 4$ [multiplicative inverse], $X = 4$ [multiplicative identity].

Obviously the purpose of an exercise such as the above is to develop a sophisticated awareness of the use of axioms, and not to facilitate efficient problem solving. The efficient problem solver would write $3X + 7 = 19, 3X = 19 - 7 = 12, X = 12 \div 3 = 4$. Unfortunately many teachers and textbooks stressed the precision of the axiomatic notation at the expense of facility with the usual notation, and basic computational and problem-solving skills suffered. The axiomatic language in this case requires more steps, and is less concise.

An example of notational improvement is taken from the APL computer language (Iverson 1966 and 1969). It is common to write max $\{a,b\}$ to denote the larger of two real numbers a and b, and a^b to represent a taken to the bth power. In APL these and many other operations are assigned special symbols, and treated as binary functions. Thus, $3 \lceil 7$ denotes the larger of 3 and 7, having the value 7; $2 \uparrow 5$ stands for 2 to the 5th power, and has the value 32. Borrowing just these symbols from APL and incorporating them into ordinary arithmetic, we see that their place in the syntax becomes the same as that of $+$, $-$, $\times$, and $\div$. Structural properties for $+$ and $\times$, such as the associative and commutative properties, can now be tested for $\lceil$ and $\uparrow$ ($\lceil$ is commutative and associative, $\uparrow$ is neither). The distributive property for multiplication across addition, which states (left distributive property) that $a \times (b + c) = (a \times b) + (a \times c)$ for all real numbers a, b, and c, can be generalized and tested for various pairs of operations: for example, $a + (b \lceil c) = (a + b) \lceil (a + c)$.

Thus the principle of using syntactically parallel notation to represent mathematically parallel concepts allows greater insight at the elementary level into the meaning of structural properties of binary functions. APL contains many other notational innovations which have potential application to the teaching of mathematics (Peelle 1974 and 1979).

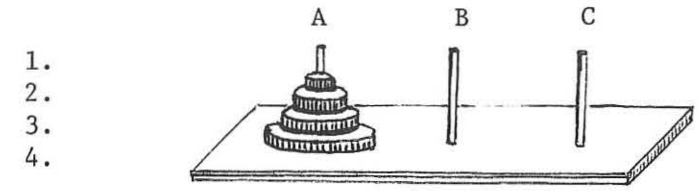
B Non-Standard Problem Representations

Sometimes a standard representation is not available — either the problem itself poses a novel symbol-configuration together with rules of procedure, as in the "checkerboard problem" above, or the solver is expected to construct a new representation for the problem. State-spaces for such problems have been used to define task structure variables, to characterize "relatedness" between problem representations, and to record the behavior paths taken by subjects (Goldin 1979). Two problems are said to be isomorphic when the states, legal moves, and solution paths of one can be placed in one-to-one correspondence with the states, legal moves, and solution paths of the other. A problem has *symmetry* if it is isomorphic to itself in more than one way.

We shall consider the example of the Tower of Hanoi problem and

its isomorphs, which have been studied by several authors (Simon & Hayes 1976; Hayes & Simon 1977; Luger 1979; Luger & Steen 1981):

4 Four concentric rings (labeled 1, 2, 3, 4 respectively) are placed in order of size, the smallest at the top, on the first of three pegs (labeled A, B, C), as in the diagram:



The object of the problem is to transfer all of the rings from peg A to peg C in a minimum number of moves. Only one ring may be moved at a time, and no larger ring may be placed above a smaller one on any peg.

The complete state-space for this problem is shown in Figure 5. Each state is labeled with four letters, referring to the respective pegs on which the four rings are located. From the network of states the problem symmetry is apparent — the roles of pegs A, B, or C can be exchanged without changing the structure of the problem. In particular, state BBBB is conjugate to the goal state CCCC, but is not itself a goal. The state-space displays forward-backward symmetry in that if CCCC is taken as the initial state and AAAA as the goal, the problem structure is unchanged.

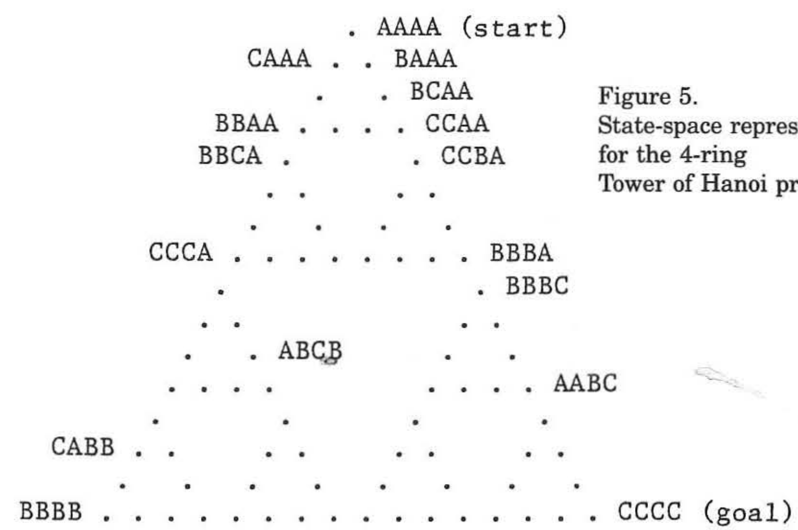


Figure 5.
State-space representation for the 4-ring Tower of Hanoi problem.

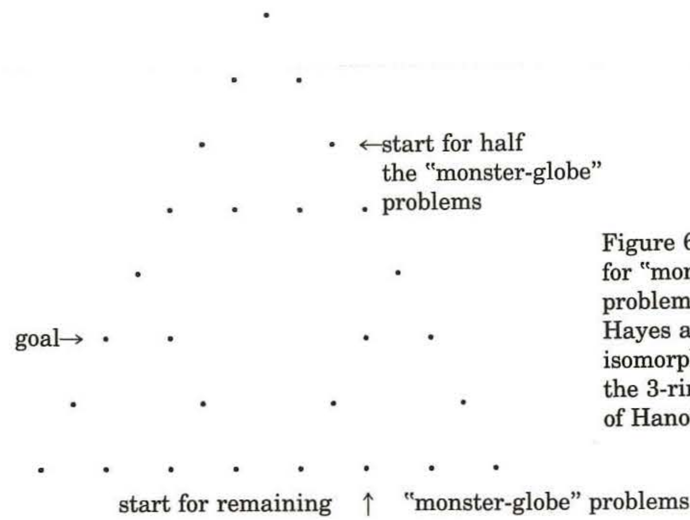


Figure 6. State-space for "monster-globe" problems of Hayes and Simon, isomorphic to the 3-ring Tower of Hanoi problem.

In the above version of the problem, studied by Luger, the pegs and the board present the solver with a notation for keeping track of moves, and solvers proceed by means of successive trials. This notation is extremely efficient for determining the availability of legal moves, but it does not preserve information about the history of moves which have occurred. The symmetry that is present is overt—"there to be noticed"—in the external representation. During the course of problem solving, some solvers who have started on a path heading "towards" state BBBB in the state-space in Figure 5 come to recognize the symmetry, and are able to correct to the symmetrically conjugate path leading to the goal. Additional discussion of symmetry as a task variable, and of overt vs. hidden symmetry, may be found in the references (Goldin 1979; Luger 1979; Goldin & McClintock 1980; Luger & Steen 1981).

Hayes and Simon employed isomorphs of the 3-ring Tower of Hanoi problem in order to study the effects of changing the problem statement on the notations adopted by subjects. The tasks were eight different "monster-globe" problems, stated in complicated language, all of which (when represented most efficiently) had state-spaces as in Figure 6. The tasks differed from each other in two ways: In Transfer problems a monster or globe was moved, while in Change problems a monster or globe changed size. Secondly, in Agent problems the monsters moved or changed the globes, while in Patient problems they moved or changed themselves. Some of the problems also differed from the others in the description of the initial state.

Since no external notational language was presented to subjects beyond the problem statement, it was necessary for them to devise their own. Three main types of notation were "invented" under these conditions, called "operator-sequence" notation, "state-matrix" notation, and "labeled-diagram" notation. These notations preserved the history of moves, but were of varying efficiency for testing the legality of moves, and not nearly so efficient as the rings-and-pegs apparatus. The types of notation remained relatively constant in frequency across problem variables. However, Transfer problems and Change problems elicited different notations within the broader categories of operator-sequence and state-matrix notations, and Agent and Patient problems elicited some differences within the operator-sequence category. For example in the operator-sequence category, indirect naming of objects was used more frequently with Change problems than with Transfer problems. In the state-matrix category Transfer problems resulted in symbols being moved from column to column, while Change problems resulted in symbols being altered within each column. Hayes and Simon postulate how the notational differences might have been caused by the differences in the problem statements.

It was observed that Change problems required greater times to solution. Change problem notations required more steps to test the problem conditions in selecting legal moves than did Transfer problem notations. This study is convincing in demonstrating how the choice of notation may affect problem difficulty through increasing the complexity of move selection.

In a discussion of the well-known "missionary-cannibal" problem, the author has suggested that the extra steps needed to test moves for legality may be described by enlarging the formal state-space to include additional "testing" moves (Goldin 1979, p. 135). In the present case this would result in a more complex state-space in which the states of Figure 6 are embedded. Such an embedding of one state-space into another is an example of one kind of state-space *homomorphism*. In general, homomorphisms may be used to describe the different kinds of relatedness which can exist between alternate problem notations.

The preceding examples lead to the following observations about efficient notational language: (1) Features of problem states which have to do with nearness to solution (as we saw in Problem 3) should be visible in the notation. (2) Problem symmetry should be overt rather than hidden wherever possible. (3) Problem representations are more efficient when the information needed to move from one symbol-configuration to the next is visually apparent in the notation, or requires few steps to obtain from each state.

IV Planning Language

This section focuses very briefly on the specialized domain of ordinary English devoted to heuristic planning. It is plausible that just as notational language can be efficient or inefficient for problem solving, so can planning language. Explicit attention to language at the planning level would then be necessary before we can teach problem solving in the same way that we now teach students mathematical notation. It may be valuable to introduce planning symbolism in order to make more visible the steps in the planning process.

Polya (1945) proposed to organize heuristic processes into four main stages: understanding the problem, devising a plan, carrying out the plan, and looking back. Much of his subsequent work was devoted to elaborating on the processes contributing to each stage. Wickelgren (1974) sought to improve problem-solving planning by introducing more technical language from artificial intelligence research — for example, he discusses "hill-climbing" which is a metaphor for state-space search algorithms with evaluation functions used in mechanical problem-solving programs. Schoenfeld (1979) devised a more elaborate stage model for organizing heuristic processes, reproduced in Figure 7.

An earlier version of Schoenfeld's model formed the basis for an extraordinarily detailed process-sequence coding scheme developed by Lucas et al. (1979), in which over fifty different symbols are used to represent process and outcome categories observed during problem solving. More recently, the author worked with Carpenter, Kulm, Schaaf, and Smith toward grouping these into a more manageable system for recording the processes used by junior high school students (Kulm, et al. 1981). This system is still undergoing revision, but in order to convey its flavor a partial dictionary is given in Figure 8. Next to each process code, the language level to which this code refers, or the translation process to which it refers, has been indicated. Thus a correspondence can be drawn between the observed processes in problem solving, and the kinds of language depicted in Figure 1.

The domain of planning language about which we can say the most, based on examples in this paper, is that which governs or talks about notational language. Silver, Branca, and Adams (1980) have examined the role of "metacognition" in problem solving. In fact, planning language as described in the present paper is a "meta-language" with respect to formal problem-solving notations. It includes the following kinds of steps: adoption of a notational language; choice of a goal or subgoal state within a notational language; modifying notational language to describe simpler problems; modifying notational language to reduce the complexity of moves; modifying notational language to

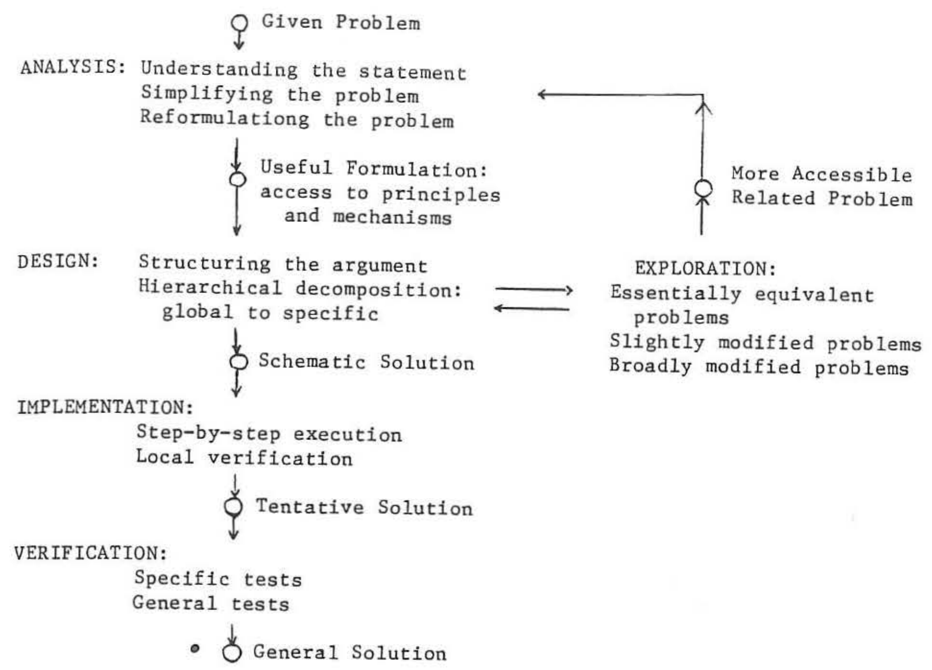


Figure 7. Schoenfeld's schematic overview of problem-solving stages (abridged from Schoenfeld 1979).

make symmetry more overt; and modifying notational language to make more visible features of problem states describing nearness to solution. In our discussion we have seen examples of how such notational modifications could greatly assist the problem solver. The inefficient or naive planner is unable or unwilling to take such steps. It therefore seems reasonable to conjecture that explicit introduction of planning language into problem-solving instruction, including practice in talking *about* problem notations and evaluating their effectiveness, could substantially improve higher problem-solving skills.

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Symbols, Icons, and Mathematical Understanding

William Higginson

Extracts are taken from the biographies of Hobbes, Rousseau, Darwin, and Russell which refer to their mathematical education. The common feature of an attraction toward geometry and an aversion to elementary algebra is noted. These experiences are analysed using theoretical positions promulgated by Davis, Hersh, Skemp, and Bruner. The central thesis is that these men probably have had difficulty learning elementary algebra because they had failed to develop a strong image or iconic representation of the concepts involved. This thesis is developed in relation to "squaring a binomial," the concept which troubled both Rousseau and Russell.

> *Mathematics is often considered a difficult and mysterious science, because of the numerous symbols which it employs.*
> *A.N. Whitehead*

Much of the power of mathematics stems from the potency of its symbols. There is, however, a price to be paid for this potency. The symbols which serve as highly effective tools for some are the most formidable of barriers for others. In the following pages a thesis is outlined which attempts to account for some of the difficulties which learners meet when studying mathematics. The method of approach is largely biographical; the essence of the argument: that we have paid too little attention to the role of images in mathematical understanding.

The unique cluster of insights, associations, and emotions which characterizes every encounter of individual with idea is never easy to capture. One of the few sources to which we can turn in such a quest is biographical literature. The examination of this literature for accounts of man meeting mathematics reveals some interesting commonalities in the experiences of a number of people. For our purposes we consider four distinguished thinkers; Thomas Hobbes (1588-1679), Jean-Jacques Rousseau (1712-1778), Charles Darwin (1809-1882), and Bertrand Russell (1872-1970).

One of the most striking features of John Aubrey's marvelous collection of short biographies, *Brief Lives,* is the picture it gives of the impact of the release of the mathematical sciences from the Greek and

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Latin tongues. Henry Gellibrand is described as, "good for little a great while, till at last it happened accidently, that he heard a Geometrie Lecture," and for Richard Stokes, M. D., we find, "His father was Fellow of Eaton College. He was bred there and at King's College. Scholar to Mr. W. Oughtred for Mathematiques (Algebra). He made himself mad with it, but became sober again, but I fear like a crackt-glasse.... Became a Sott." Few entries can compare for vividness, however, to the one for the philosopher Hobbes where his first exposure to geometry is noted.

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 El. libri I. He read the Proposition. By G–, sayd he, (he would now and then sweare an emphaticall Oath by way of emphasis) this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.
>
> I have heard Mr. Hobbes say that he was wont to draw lines on his thigh and on the sheetes, abed, and also multiply and divide. (p. 309)

Mathematical ideas play a very limited role in Rousseau's spirited autobiographical *Confessions*. There are, however, two passages which are of interest. The first perhaps tells us something of eighteenth century attitudes about mathematicians. In 1744 Rousseau had a short and less than satisfactory liason with a sultry Venetian courtesan called Giulietta. At their parting the vengeful young woman advised Rousseau in a cold and scornful voice to "Give up the ladies and study mathematics" (p. 302). The other passage is more appropriate for our purposes for in it Rousseau describes some of his mathematical education as follows:

> I have never been sufficiently advanced really to understand the application of algebra to geometry. I disliked that way of working without seeing what one is doing; solving a geometrical problem by equations seemed to me like playing a tune by turning a handle. The first time I found by calculation that the square of a binomial figure was composed of the square of each of its parts added to twice the product of one by the other, despite the fact that my multiplication was right I was unable to trust it until I had drawn the figure on paper. It was not that I had not a great liking for algebra, considered as an abstract subject; but when it was applied to the measuring of space, I wanted to see the operation in graphic form; otherwise I could not understand it at all. (p. 3)

Charles Darwin's massive contribution to the intellectual life of the nineteenth and twentieth centuries would most likely have come as a considerable surprise to his youthful contemporaries, for his scholastic record as a schoolboy and undergraduate was far from prepossessing. In 1847, sixteen years after receiving his B. A. from Cambridge, Darwin wrote in a letter to his friend Hooker, "I am glad you like my *Alma Mater,* which I despise heartily as a place of education." In another passage in his autobiography we find:

> During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. (p. 18)

Darwin's son Francis, who edited the *Autobiography and Selected Letters* of his father, notes in another passage:

> My father's letters to Fox show how sorely oppressed he felt by the reading for an examination. His despair over mathematics must have been profound, when he expresses a hope that Fox's silence is due to "your being ten fathoms deep in the Mathematics; and if you are, God help you, for so am I, only with this difference, I stick fast in the mud at the bottom, and there I shall remain." Mr. Herbert says: "He had, I imagine, no natural turn for mathematics, and he gave up his mathematical reading before he had mastered the first part of algebra, having had a special quarrel with Surds and the Binomial Theorem. (p. 114)

It is perhaps the passion which most catches one's attention in these passages. One might well expect, for instance, that men whose contributions were to non-mathematical fields should have had certain difficulties with the discipline. It is, however, something more of a surprise to find that one of the finest logico-mathematical minds of the last hundred years experienced difficulties with elementary algebra almost identical to those of Rousseau. In his autobiography (1968) Bertrand Russell writes:

> The beginning of Algebra I found far more difficult, perhaps as a result of bad teaching. I was made to learn by heart: "The square of the sum of two numbers is equal to the sum of their squares increased by twice their product." I had not the vaguest idea what

this meant, and when I could not remember the words, my tutor threw the book at my head, which did not stimulate my intellect in any way. (p. 34)

This difficulty was to prove a temporary one: "After the beginning of Algebra, however, everything else went smoothly" (p. 34). And some seventeen years later Russell would be embarking on what has been called "the longest chain of deductive reasoning that has ever been forged" (*Spectator,* p. 142). This, of course, was *Principa Mathematica,* the three-volume treatise on the foundations of mathematics which Russell co-authored with Alfred North Whitehead. The anonymous reviewer in the *Spectator* wrote of a work which "may be said to mark an epoch in the history of speculative thought" but went on to observe:

> It is easy to picture the dismay of the innocent person who out of curiosity looked into the later part of the book. He would come upon whole pages without a single word of English below the headline; he would see instead, scattered in wild profusion, disconnected Greek and Roman letters of every size interspersed with brackets and dots and inverted commas, with arrows and exclamation marks standing on their heads, and with even more fantastic signs for which he would with difficulty so much as find names. (p. 142)

We wish to set our analysis of these biographical excerpts in the context of the view of four theoreticians: the mathematicians Davis and Hersh on the role of symbols in mathematics, the mathematics educator Skemp on types of mathematical understanding, and the cognitive psychologist Bruner on modes of symbolic representation.

In their recent book Davis and Hersh (1980) have a section entitled "Symbols" in which they observe:

> What do we do with symbols? How do we act or react upon seeing them? We respond in one way to a road sign on a highway, in another way to an advertising sign offering a hamburger, in still other ways to good-luck symbols or religious icons. We act on mathematical symbols in two very different ways: we calculate with them, and we interpret them.
>
> In a calculation a string of mathematical symbols is processed according to a standardized set of agreements and converted into another string of symbols. This may be done by machine: if it is done by hand, it should in principle be verifiable by a machine.
>
> Interpreting a symbol is to associate it with some concept or mental image, to assimilate it to human consciousness. The rules for calculating should be as precise as the operation of a computing machine: the rules for interpretation cannot be any more precise than the communication of ideas among humans. (p. 121)

Skemp (1976) has made a distinction between "instrumental" and "relational" understanding which has proven to be useful in analysing situations in mathematics education. In an instrumental approach to the teaching of mathematics major emphasis is placed on the acceptance and application of definitions and rules. The questions of why one would want such definitions and how the particular rules come into being are not appropriate in the instrumental approach. They are, however, essential features of the relational approach.

One of the most complete theories about the nature of the symbolizing process is the one developed by Jerome Bruner and his co-workers over a number of years (1966, 1968, 1973). Bruner (1973) distinguishes three modes of representation — the enactive, the iconic, and the symbolic:

> Their appearance in the life of the child is in that order, each depending on the previous one for its development, yet all of them remaining more or less intact through life.... By enactive representation I mean a mode of representing past events through appropriate motor response.... Iconic representation summarizes events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images. Images stand for perceptual events in the close but conventionally selective way that a picture stands for the object pictured. Finally, a symbol system represents things by design features that include remoteness and arbitrariness. A word neither points directly to its referent here and now, nor does it resemble it as a picture. (p. 328)

An analysis of the biographical statements of the four individuals in question reveals two strong common underlying themes. The first is that of an attraction, often a passionate one, for geometry. We have noted Hobbes' addiction, and Russell's feelings were no less strong. He writes, for instance, "At the age of eleven, I began Euclid with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world" (1968, p. 33).

Even the algebraphobic Darwin had enjoyed geometry. "Again in my last year I worked with some earnestness for my final degree of B. A., and brushed up my Classics, together with a little Algebra and Euclid which latter gave me much pleasure as it did at school" (1958, p. 19).

Rousseau too was attracted to geometry, but he makes a significant qualification. "I went on from there to elementary geometry.... I did not like Euclid, who is more concerned with a series of proofs than with a chain of ideas; I preferred the geometry of Father Lamy, who from that time became one of my favourite authors, and whose works I still re-read with pleasure" (1970, p. 226).

The second commonality is a strong dislike for situations in elementary algebra where it proved difficult to attach any meaning or imagery to the manipulation of symbols. We have seen the views of Rousseau, Russell, and Darwin. Aubrey writes of Hobbes in this connection: "He would often complain that Algebra (though of great use) was too much admired, and so followed after, that it made men not contemplate and consider so much the nature and power of Lines" (p. 309).

With the theoretical positions sketched previously in mind we can set the experiences of Hobbes, Rousseau, Darwin, and Russell in a somewhat more general context. What we have is perhaps not so much a difference between algebra and geometry as branches of mathematics, as rather a situation where the algebra is learned instrumentally, and the geometry relationally. The nature of geometry is such that it lends itself easily to the production of images. This is not so clearly the case with algebra. It is possible, of course, (note the case of Rousseau) to teach geometry instrumentally as well, with the same negative results.

To recapitulate: we see in these four cases, examples of what is probably a very common phenomenon, the presentation of mathematical ideas almost entirely in the symbolic mode of representation. The result of this is that learners fail to have any significant understanding of the situation. Equivalently, using terms which accentuate the iconic nature of their difficulties, they lack *insight* or fail to *see* what is going on. The possibility of remedying this situation by consciously constructing icons for mathematical symbols is an obvious one.

We consider as an example the Rousseau/Russell problem of $(a + b)^2$, the squaring of a binominal. From their descriptions it would seem that both men were encouraged to learn this concept instrumentally. The "rule" is that "the square of the sum of the numbers is the sum of their squares increased by twice their product," that is, $(a + b)^2 = a^2 + b^2 + 2ab$. (An inventive mind like Russell's was able to put even the most arcane bits of knowledge to use. Further on in his autobiography we find, "I used, when excited, to calm myself by reciting the three factors of $a^3 + b^3 + c^3 - 3abc$; I must revert to this practice. I find it more effective than thoughts of the Ice Age or the goodness of God" [1970, p. 38].) Expressed only in this way, this mathematical idea seems little better than a variation on tongue twisters of the genre "Peter Piper picked a peck of pickled peppers."

or more $12 \times 12 = (10 + 2)(10 + 2) = 100 + 20 + 20 + 4 = 144$
generally $12 \times 12 = (11 + 1)(6 + 6) = 66 + 66 + 6 + 6 = 144$
and $8 \times 16 = (2 + 6)(13 + 3) = 26 + 6 + 78 + 18 = 128$

Figure 1.

Yet this need not be, since this is nothing more in some senses than a compact way of noting an infinite number of arithmetic statements of the sort listed in Figure 1; this fact seems seldom to be mentioned in textbooks on elementary algebra.

An even more surprising omission from these books is an obvious iconic representation for the symbolic statement. (It is almost certainly the same one which gained Rousseau's trust.) This icon hinges on the fact, so central to Greek mathematics, that just as we can associate the sum of two positive integers with a particular line segment, we can associate their product with the area of a rectangular figure. If we have two positive integers a and b, a powerful image of the square of their sum is a square of dimension $(a + b)$. As can be seen from Figure 2, this large square is composed of four rectangles: a square of side a, a square of side b, and two identical rectangles of area ab.

Once opened, this door leads to many other related mathematical ideas; for example: the square of a trinomial $(a + b + c)^2$; the cube of a binomial $(a + b)^3$; the difference of two squares $(a^2 - b^2)$; Russell's factoring problem; the square root of two, i.e., find b so that $(1 + b)^2 = 2$. It should probably be noted, as well, that this area is a critically important one in mathematics. It took the genius of a Newton to fully generalize this situation to the case of $(a + b)^n$ and this result was a key step in the development of the calculus. It is not reassuring to observe that we have in many ways progressed very little from the time of Rousseau as far as the teaching of this concept is concerned.

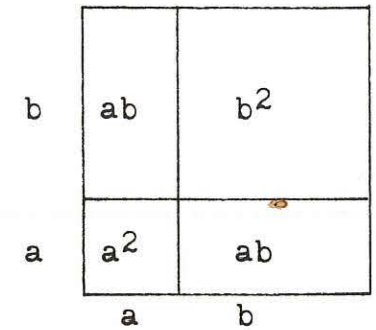

Figure 2.

$$(a+b)^2 = (a+b)\,(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

F O I L

Figure 3. The expansion of a binomial by the FOIL law. The product of the sums is equal to the sum of the products of the first, outside, inside and last terms.

The most popular technique for explaining the product of two binomials in many parts of North America at present is the "FOIL Rule," a blatant appeal to authority (see Figure 3). (There are those who would contend that the whole purpose of teaching mathematics in schools is to have children learn how to accept authority, very often in forms which seem irrational, meaningless, and arbitrary. However, that is another, albeit very important, issue.)

But how typical is our example? Is it possible that only some mathematical concepts have iconic representations or can one legitimately expect to find images playing an important role in the development of all mathematical ideas? It must be acknowledged that the idea of a 'mental image' is one that has been hotly debated in a number of disciplines over the years. Philosophers, psychologists, artists, and mathematicians have all, at one time or another, participated in the fray (Arnheim 1972, Gombrich 1959, Hannay 1971, Mason 1980, Paivio 1971, Plato, and Wertheimer 1968) and there are few principles in the area which would gather universal acceptance. Hadamard's classic work in the field (1954) would seem to show unequivocably that imagery is of critical importance in the thought of creative mathematicians. He quotes Einstein, for instance, as saying "The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined" (p. 142). More recently we have the report of the mathematician Papert (1980) who tells of the significant role played in his intellectual development by the image of gears. In general, however, as Bruner (1973) notes, the situation is that "we know little about the conditions necessary for the growth of imagery and iconic representation" (p. 329).

Any conception of mathematical understanding which emphasizes the iconic representation of concepts must acknowledge its roots in the

thought of the ancient Greeks. (In his "On Memory and Recollection," Aristotle contended that it was "impossible even to think without a mental picture"). Iverson (1972, 1980) and Hammersley (1979) have written about the limitations of our contemporary mathematical symbols and have made suggestions as to where improvements might be made. Looking to the future it seems obvious that there is great potential for the graphic representation of mathematical ideas through the medium of computers (Papert 1980). It remains to be seen whether or not we will be able to make mathematical symbols more understandable with the aid of computers. In the meantime it is of interest to note that 190 years ago, Samuel Taylor Coleridge, then seventeen years old, observed in a letter to his brother George:

> I have often been surprized, that Mathematics, the Quintessence of Truth, should have found admirers so few and so languid — Frequent consideration and minute scrutiny have at length unravelled the cause — Viz — That, though Reason is feasted, Imagination is starved: whilst Reason is luxuriating in it's proper Paradise, Imagination is wearily travelling over a dreary desart. (p. 7)

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Towards Recording

Nick James and John Mason

Behind the formal symbols of mathematics there lies a wealth of experience which provides meaning for those symbols. Attempts to rush students into symbols impoverishes the background experience and leads to trouble later. In conjunction with manipulating objects it is essential to provide time for talking about their activities and developing their own informal records before meeting the formal symbols of adult mathematicians. We present three examples of children's work which demonstrate these steps in the struggle to move towards recording perceived patterns.

To most people the formal symbols used in mathematics seem cold and lifeless. Even the ubiquitous x is literally an unknown quantity with little meaning. Mathematicians often seem content to lend credence to this view by talking about mathematics as a formal game. This view of mathematical symbols is misleading in its incompleteness. In fact, for symbols of any kind to be of value there must be a wealth of background experience which can be called upon. This article is concerned with developing that rich background experience in the important phase of mathematical learning/investigation which we call TOWARDS RECORDING. The struggle to capture an insight which is as yet pre-articulate is often overlooked in a rush to lead students into formal symbols, resulting in an impoverished if not empty background experience, and producing frustration, anxiety, and math-phobia. We present three examples of childrens' work when time was taken to let the children participate in the struggle towards recording. The results reveal something of the stages in that process.

Keith and Ranjit (age 12½)

Task A Choose a rod and make up the equivalent length using repeating patterns (an example was given). Figure 1 is but one of the many pattern sets made by the children.

Task B Talk about the rods in each row in as many ways as possible (several variations were discussed). See Figure 2.

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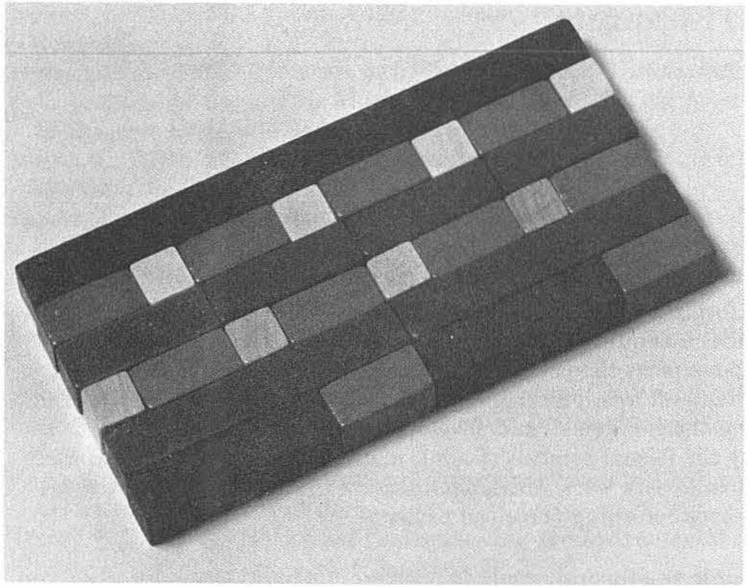


Figure 1.
Color of rods down left side:

purple rod
pink rod
light blue rod
white rod
dark blue rod
red rod

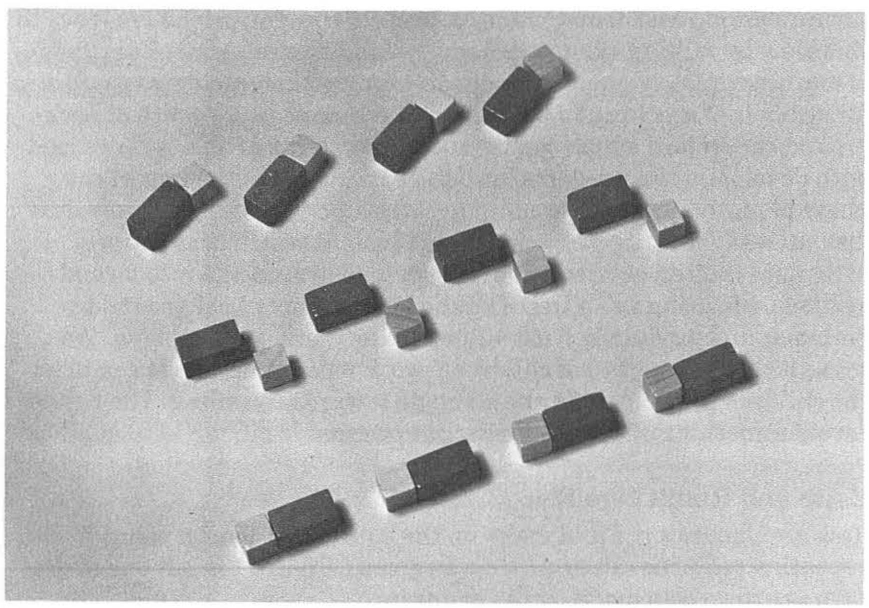


Figure 2. Discussion led to the second row from top being described as
"Pink-and-white . . . (pause) . . .four times"
"Four pink and four white"
"Four. . . (pause) . . . white-and-pink"

1ST ROW KEITH AND RANJIT
= 4 pink and White = Purple Rod
= 4 pink and 4 white = " "
= 4 pink + white x 4 = " "
= Pink + White x 4 = " "
= Pink, white, pink, white, pink, white, pink white
= 4 white and Pink
= 1 pink plus a white x 4
= 4 pink + white joined together
= Pink + white x 4

Figure 3.

Task C Write down the various ways of talking about the rods. See Figure 3. The teacher then asked them questions such as, "If I read Pink + White × 4, what rods would I put out?" All agreed on one pink followed by four whites. The attention was drawn to the need for agreement and precision. The crucial task came in a subsequent session.

Task D What must you write to leave no doubt about which rods you mean the reader to put out. Explore this amongst yourselves. Figure 4 shows what Ranjit wrote.

Purple Rod =
4 Pink and 4 white joined together ~~x4~~
= 1 Pink and 1 white joined together x 4
Pink, white x 4 = 4 Pink and 4 White joined together

Figure 4.

Sensing that Keith and Ranjit had struggled to express an as yet pre-articulate sense of brackets, the teacher then offered Figure 5.

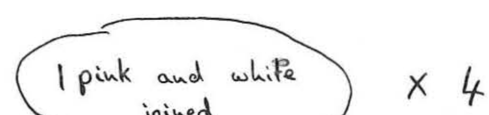


Figure 5.

Figure 6.

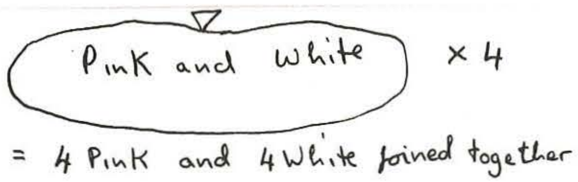
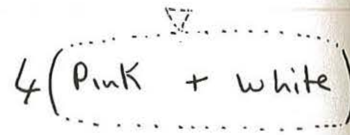


Figure 7.



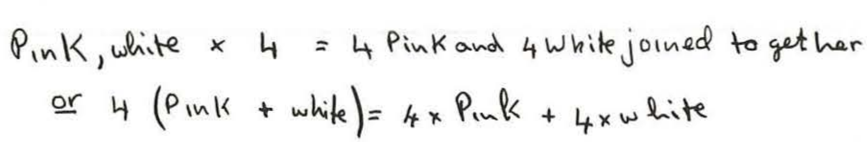
Further discussion elaborated this to the idea of four bags each containing a pink and a white, and drawings like Figure 6 were adopted into their records. Later the balloon or bag was partly erased to yield the more common bracket (Figure 7). Furthermore, the children had begun to get a grasp of the distributive law (Figure 8). Further experience of this activity and also others involving the distributive law helped Keith and Ranjit to establish a strong sense of the use of brackets and the meaning of the distributive law. This is what is meant by a wealth of experience supporting the distributive law. The same process emerges in the next two examples which are based on investigations presented to two different age groups.

Lesley (age 9½) in a group of six children

Task A With 17 interlocking cubes make a square picture frame. All the while the children talked among themselves: "Two pillars of five down the side and a band of three along the top and bottom." After many trials they finally decided that it was impossible to use all 17 cubes to make a square picture frame. The teacher then asked them about other frames and after much discussion, other examples were produced (Figure 9).

Task B How many cubes are needed to make larger frames? Record your results! Figure 10 shows Lesley's results. Notice that the compelling nature of the underlying pattern has deflected Lesley from answering the original question. This will emerge if Lesley is invited

Figure 8.



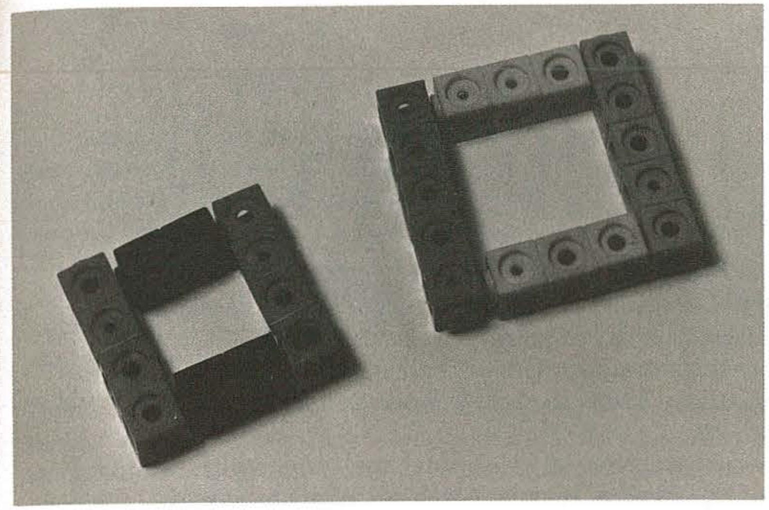
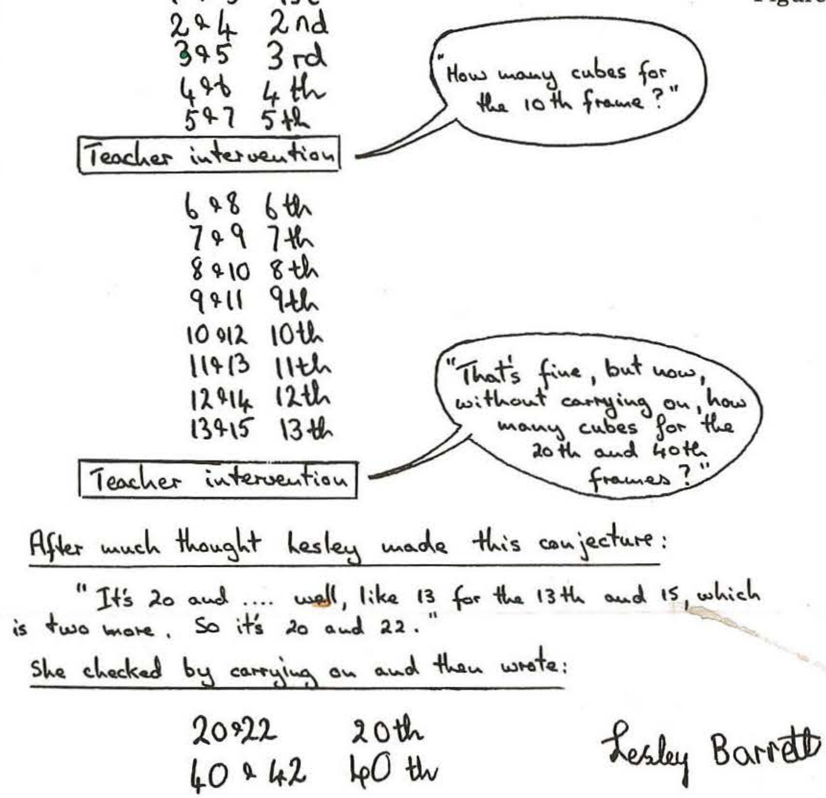


Figure 9.

Figure 10.



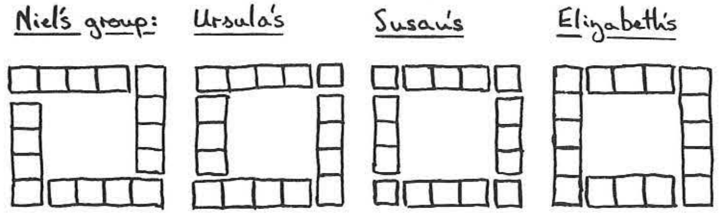


Figure 11.

to check her resolution against the task set. Time spent recording results is well rewarded though. In this case Lesley has come to her generalization from the support of her neat and systematic record. Unfortunately most adults are very reluctant to record results and consequently they find it harder than necessary to come to generalizations.

Patterns such as picture frames which invite generalizations lie at the heart of pre-algebra, because algebraic symbols arise in response to a need to record a generalization succinctly. The next task sequence shows how this can happen.

From picture frames to algebraic expressions

Task A Explore the range of possible picture frame designs to surround a 3 × 3 photograph. The children got together in groups and spent fifteen minutes producing a series of rough sketches drawn on squared paper. Much discussion of the possibilities ensued. Some of the sketches produced are shown in Figure 11. Each group in turn then described their designs to the rest of the class. Before attempting to generalize their designs to cover frames for square photographs of any size they need further experience of some other special cases. The teacher then suggested:

Task B Decide on a particular design and produce a whole range of picture frames, including the next smaller and the next larger ones. Figure 12 shows what Susan's group produced.

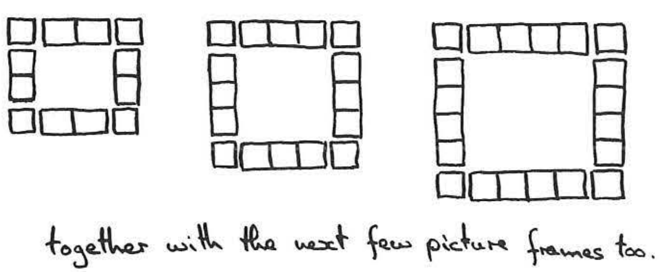


Figure 12.

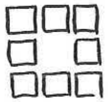


Figure 13.

Discussing their range of frames, Susan's group eventually added the smallest frame of all (Figure 13) saying: "This must be part of our collection. It's got four squares in the corners just like all the others. It doesn't have long bars in between though. They're only one long, but it's built the same as all the rest." They even paused to discuss the case of one four square at each corner with bars of length zero in between! This was finally rejected because such a frame couldn't contain a picture.

Task C Can you say how many cubes you'd need to make a frame to surround a square picture of any given size? Ursula's group offered this attempt at a generalization: "You must leave one square in the corner. On the left you have a rod the same size as the side of the picture in the centre. Then for the other three you just add one onto the size. You've got three of those." Once again the pattern Ursula's group was using to make each of their picture frames deflected them from answering the question in Task C.

Task D Write down how you'd find the number of cubes needed to make a frame for a square picture of any given size. Figure 14 shows what Ursula's group wrote. The teacher asked what they meant by "size of the picture" and they explained that it was whatever size the person happened to be thinking of. "If it was a 100 × 100 square photo then what would it be?" the teacher asked. The group explained: "One, for the corner; plus 100, for the size; plus (101) × 3 for size + 1 taken 3 times; that's 404."

The children's explanation suggested to the teacher the idea of a "thinks cloud," like those used in the comics, to represent "the size of the picture I'm thinking of." He offered the cloud idea to the whole class as a means of refining the rather wordy generalizations they were all producing. He drew a stick person with a "thinks bubble" coming from its head to illustrate his point.

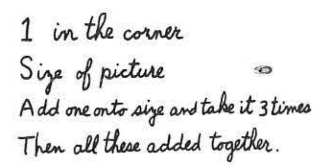


Figure 14.

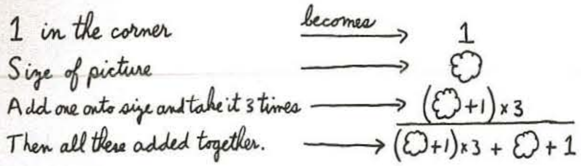


Figure 15.

Task E Try and shorten your statements using the notion of a "thinks cloud." Ursula's group added arrows to what they had already written pointing to the cloud notation (Figure 15). Having negotiated this shorthand with the class, it was a relatively short step for the teacher to introduce the more usual algebraic notation. "Whilst everyone in this class knows what we mean by 'cloud,'" he said, "the world's mathematicians use a letter like 'n' to stand for 'the number we're thinking of.'" Ursula's cloud shorthand was readily turned into Figure 16. The same process of recording and refining was carried out for the other groups (Figures 17 and 18).

the number of cubes needed to surround a square picture of side n is:

$$3(n+1) + n + 1$$

Figure 16.

The purpose of these examples has been to indicate the delicacy of the period leading up to adoption of a standard symbol system. Building on children's experience of doing specific tasks with apparatus, diagrams, or previously mastered symbols and depending on attempts to articulate to each other and to the teacher what they are doing, the act of moving towards recording is what gives substance to the otherwise heartless symbols of mathematics. Crucial aspects in order for the struggle towards recording to be meaningful include: (i) Minimal teacher intervention except to plant the seeds of helpful language patterns and recording devices, (ii) Sufficient time for the children to get to grips with the task, (iii) A minimum of three distinct instances from which a generalization can be formed, (iv) Encouragement of children to do tasks and talk about what they are doing, (v) A neutral environment permitting children to make conjectures which may be modified, without the stigma of being right or wrong.

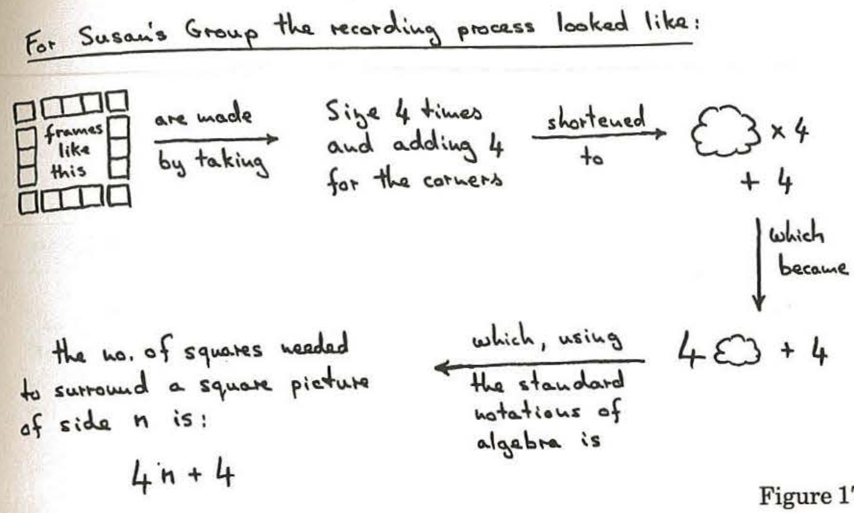


Figure 17.

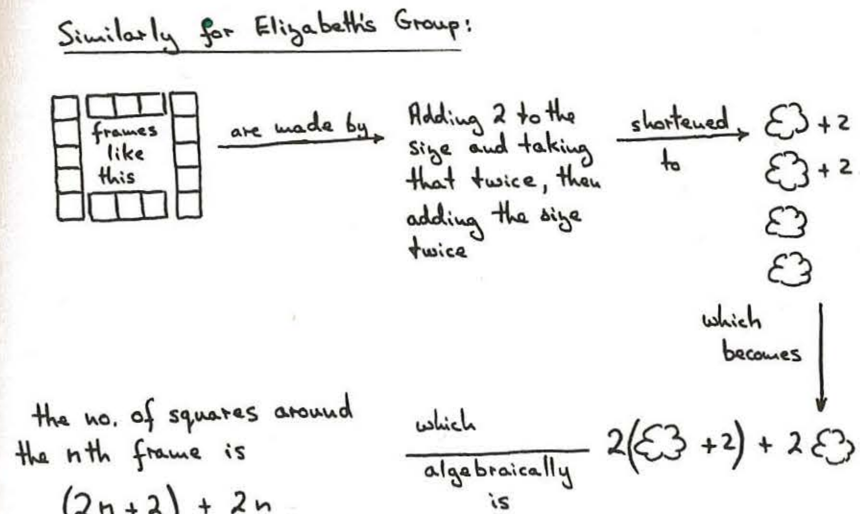


Figure 18.

The activities of doing, talking, and recording are classroom activities which facilitate the corresponding shifts in psychological states described in Mason (1980), moving from

Enactive to Iconic, that is from confident manipulation of specific instances to getting a sense of a common generalization;

Iconic to Symbolic, that is articulating the sense of generalization as a sequence of conjectures which are modified until they crystallize into an articulate and recorded statement which captures the notion.

The transition from Symbolic to Enactive, that is from an abstract form which is constantly referred back to examples to recall its intention, to a confidently manipulable entity which can serve as a component in a new, higher order notion,

requires practice to achieve mastery of the symbols. This is the true role of exercises in the mathematics classroom.

Reference

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Mental Images and Arithmetical Symbols

L. Clark Lay

Experiments by psychologists have led to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. An analysis is begun of the mental images that are judged to be properly evoked by certain number symbols of arithmetic. A variety of graphical models are suggested for use in linking these symbols to the desired mental construct. Some of these models have been found to be advantageous and may prove to be critically essential in certain mathematical contexts. Their assets and liabilities are discussed, and suggestions are made for modifications of conventional curriculum practice. A rich field of investigation exists in the visual imagery that can be associated with elementary mathematics. Progress here holds promise of extending mathematical competence to a larger portion of society.

The role of imagery in human thought has been studied by Piaget and Inhelder (1971), particularly as it relates to Piaget's well known genetic model of intellectual development. Their experiments led these authors to the conclusion that images play an indispensable, if subordinate, role in thought as symbols. In our paper an analysis is begun of the mental images that are judged to be properly evoked by certain symbols of arithmetic. The emphasis will be on graphical models that can be used to link such symbols to the desired mental construct.

An experiment

The reader is invited to join in the following experiment. Writing materials such as a pencil and paper should be available. In a moment you will be presented with a very familiar symbol. You are asked to respond to this symbol, in the following manner:

Imagine yourself giving a verbal explanation of the meaning of this symbol to a person for whom it is not as yet familiar. Assume that the verbal discussion has not gone as well as you had hoped, and that it has occurred to you that a sketch or diagram of some sort might be helpful. You are asked to show your choice for this purpose. It is of particular interest that you record the first image that comes to your mind when this symbol is presented. If, upon further reflection, you

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can think of other sketches you might use, we will be interested both in their variety and in the order of their coming to your mind.

Ready? The symbol to which you are to respond is "5"; the numeral for the number five. What image did 5 first evoke in your mind?

As an alternate experiment, the word five can be given orally, although I have not found this to affect the results to a significant extent. For the past many years the author has tried this experiment with subjects of wide diversity of attainment in mathematics, ranging from primary school pupils to university graduate students. When the study is limited to the initial response, there has been a uniform consistency in the type of diagrams that are drawn.

With very few exceptions the image that seems first to come to mind is that of an array of five separate but similar objects. These may be just five vertical lines, | | | | |, or these may be tied together as the tally , or there may be an arrangement in a characteristic pattern such as for a domino, * * * * * . Other subjects may show the fingers and thumb of one hand, or they may represent a collection of recognizable objects such as flowers, apples, or rabbits.

It would seem that even for those who have acquired a considerable sophistication in mathematics the symbol 5 is first perceived in its relation to counting as enumeration. But there is a considerable variety of ways to think of five. Some of these are not only advantageous but may even be critically essential in certain mathematical contexts. And these situations need be no more complex than those commonly introduced in the elementary schools. A list of twelve such representations of the number five appears at the close of this paper. These will be discussed in turn.

Numbers as counters

The first five letters of the English alphabet can be listed as; a,b,c,d,e. The acceptance of this collection of letters as a single whole can be aided by enclosing the given list by braces, { }. A temporary name, such as the letter S, can then be assigned to this collection, or set, of letters. Let #( ) be an operator; a symbol which directs that the number of members in the set be determined by counting. We then say that the number five is thus represented as the cardinal number of a set.

$S = \{a, b, c, d, e\}; \#(S) = 5$

During the mathematics education reforms of the 1960's this set representation of numbers was widely advocated, even for the

first introduction to number concepts. For various mathematical investigations, particularly those at an advanced level involving infinite sets, the advantages of set language and symbolism had already become widely known and accepted. It was hoped that the use of these mental models might also be an enlightening experience for the young learner as well.

But trials failed to support this innovation. Indeed, after a time, "sets" became almost a synonym for "what's wrong with the new math?" In retrospect it can be seen that set representations were tools that were too delicate for the tasks assigned to them; there were too many niceties to be observed in their use; so that confusion was often increased rather than decreased.

As an example of the care that must be taken, note that one needs to differentiate between a list and a set. Thus the list *a,a,a,* is different from the list *a,a,* and from the single listing, *a*. But, going back to the set S above, the notation used is defined as a roster notation: that appearing between the braces is a listing of *names*. But names used in this manner must be distinguishable; a repetition of the same name would introduce ambiguity. Hence $\{a,a,a\}$, $\{a,a\}$, and $\{a\}$ must then all be accepted as representing the same set.

To return to number as represented by an array of counters, we can anticipate trouble with the number zero. For centuries people must have thought: If there are no objects to be counted, what is the need of a number for this situation? Menninger (1969) found no trace of a written symbol for zero earlier than a Brahmi inscription of AD 870, although he states that the Sanskrit language had a name for this idea in *sunya* (empty) in the sixth century, and that the astronomer Ptolemy (about AD 150) used an abbreviation of a Greek word to indicate a missing place when writing fractions of Babylonian origin. Dantzig (1941) conjectured that zero was first conceived by an ancient scribe who wished to record an empty column on his counting board. Menninger puts it this way: "The zero is something that must be there to say that nothing is there."

If, as suggested, most persons associate numbers very strongly with the counting of objects, it is understandable why zero is often known only as nothing (no-thing). The set representation of numbers introduces the empty set as a model for the number zero. But again, it is just too easy for the beginner to confuse the emptiness (no-things) of this set with the set itself; since the empty set is only a convenient mental fiction, but nevertheless must be considered to be something (some-thing).

Counting of changes

The number zero has a much improved status when *changes* rather than objects are counted. Events can be considered as changes of state. Zero is then the number assigned to the original or initial state; before any of the changes to be counted have taken place. For such counting, zero is no longer tied to an absence of things, but rather to a lack of change.

In Figure 1 (adapted from Lay 1977) contrast is shown between the counting of objects (above the line), and the counting of changes (below the line). With the latter, five is now represented by a counting sequence. The arcs between the numerals for this counting sequence are meant to suggest changes of any kind that take place. A number is not assigned to the change while it is happening; the count is recorded only after the change is complete.

Counting objects	0	1	2	3	4	5
		*	*	*	*	*
Counting changes of state	0 ⌒	1 ⌒	2 ⌒	3 ⌒	4 ⌒	5

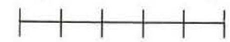
Figure 1.

There is a wealth of familiar activities and experiences which provide reason for counting changes of state. A simple example would be the counting of changes of position, as by steps. Zero then designates the starting position, the number 1 is recorded after the first step, 2 after the second, and so on. These further observations can be made for the comparison between the counting of objects (above the line) and the counting of changes (below the line).

zero	none, no object
	initial state, origin
one	object
	change, transformation
+1, unit increase	join one object
	advance to the next state
−1, unit decrease	remove one object
	return to the previous state

The concept of a counting sequence was used by Dedekind (1888) and Peano (1889) in their developments of a logical foundation for the principles of arithmetic. Such sequences are based on very fundamental intuitions. The questions to be answered are: "What comes first?" and then repeatedly, "What comes next?" A child is beginning to grasp these ideas when he or she can repeat, "Mary had a little lamb." Figure 2 suggests some of the significant reorientation that must take place when the imagery for numbers is to be shifted from counting to measuring.

Figure 2. From counting to measuring

* * * * *	
1 2 3 4 5	0 1 2 3 4 5
Counters	Scale
Counting	Measuring
How Many?	How Much?
Multitude	Magnitude
Separate	Connected
Discrete	Continuous
Natural Numbers	Non-negative Real Numbers

A small proportion of the persons who have participated in the thought experiment for numbers, as previously discussed, have sketched a scale for the number 5, similar to that in Figure 2. But most have not thought of doing this, even when encouraged to do so by leading questions.

One disturbing fact has come out of verbal discussions of such simple scales. There are persons who believe that Figure 3 is really a model for number six, rather than for five!

Figure 3.

Apparently they are so committed to the counting of objects that they react by counting the scale division points, rather than counting the line segments, or in thinking of the measure of the length of the entire segment. School authorities recognize the widespread avoidance by pupils of all the physical sciences, because of the reputed difficulty of these subjects. Much of the data for these sciences comes from measurements of quantities which the mind conceives as being continuous; such as mass, time, and measurements in space. What is the barrier to success for beginning students in these sciences? Can it be partially attributed to the pupils' lack of appropriate mental images for the symbols they encounter?

Models for rational numbers

The first extension of the number system, beyond that for the counting numbers, has traditionally been to the non-negative rationals, commonly known as fractions. Let us repeat the symbol response test, this time for the fraction, two-thirds.

What type of sketch or diagram first comes to your mind as being useful to communicate the meaning of $2/3$?

It can be anticipated that nearly all people will first draw a unit of some kind; it's "oneness" being suggested by its appearance of being "all there." Examples might be a circle, or a pie, or possibly a square figure. This is then divided in three parts of equal size, and the attention is directed to two of these subdivisions, by some device such as shading. For a verbal description we may say that two-thirds has thus been shown as representing two of the three equal parts of one (one unit, or all of something). But the fraction $2/3$ also represents one of the three equal parts of two, although a figure to illustrate this interpretation is very rarely given by subjects for our experiment. If the two is imagined as referring to two separate objects, this figure has a forbidding aspect if one is contemplating dividing it into three equal parts. This should be compared with the ease of *thinking* about a length (with a measure of two units), and sub-dividing this into 3 parts of the same length.

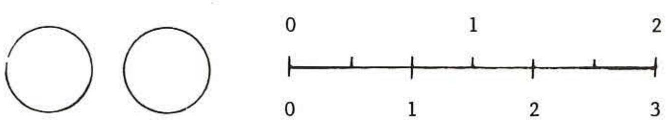


Figure 4.

The key strategy here is to take advantage of the arbitrary length that can be assigned to the measure of one unit. We begin with a line segment with designated points that are equally spaced. The zero and 1 of the scale are then located so that this assumed unit length can readily be subdivided into the prerequisite number of parts. With this done, then any positive integral multiple of this chosen unit length can be easily divided into the same number of integral parts. Thus in Figure 4 the number 1 was located to show the unit length divided in three parts; this assured that the length with measure 2 could also be so divided. There is a striking difference in the conceptual difficulty of thinking about dividing a two foot length of string into three equal

lengths, as compared to thinking about dividing two apples into three equal portions. If this example with $2/3$ does not seem sufficiently impressive, one need only try contrasting the discrete and continuous models using slightly larger numbers, such as $3/7$. An important pedagogical advantage can be noted for the models using scales. A variety of illustrative examples are easily constructed by pupils, who are the ones who need the practice. But for the models for which the units are separated, both text and teachers are limited to the simplest of cases.

When the symbol x/y is interpreted as x of the y equal parts of one, this concept is commonly termed the parts-of-a-whole meaning. When x/y is thought of as the measure of one of the y equal parts of x units, we are appealing to the quotient meaning for x/y. Of these two interpretations, the quotient meaning appears much more frequently in applications but seems to be far less familiar to most adults. My hypothesis is that this handicap is strongly associated with the lack of the mental imagery that visualizes numbers as measures, such as their use on a linear scale.

The symbol x/y has still another interpretation, and its application extends to an even broader field than the two meanings already mentioned: The symbol x/y is also used to represent the ratio comparison of x to y. In part A of the Figure 5 we have a model for thinking about how 2 compares to 3. If a difference comparison is used (by subtraction), we say that 2 is 1 less than 3. But with a ratio comparison (by division), we say that 2 is $2/3$ of 3.

This same ratio comparison of 2 to 3, or of $2/3$, is also shown by diagrams B and C. Of the three, diagram C is considerably more flexible in its application. This flexibility arises from this distinctive property of ratio comparison: The ratio comparison of two magnitudes is independent of the scale used to measure them. Not only is C of Figure 5 a representation for the ratio meaning of $2/3$; it serves equally well for $2{,}000/3{,}000$ and for $.02/.03$, as well as $1/1\frac{1}{2}$ and $1.6/2.4$. An older notation for the ratio of 2 to 3 was 2:3, but there is increasing use of the same form as for fractions and quotients.

Scales for the measurement of length need not be confined to straight line segments. They are also used with curved figures, in particular with arcs of circles. Many phenomena in life are cyclic in nature; the same succession of events is repeated over and over again.

Figure 5.

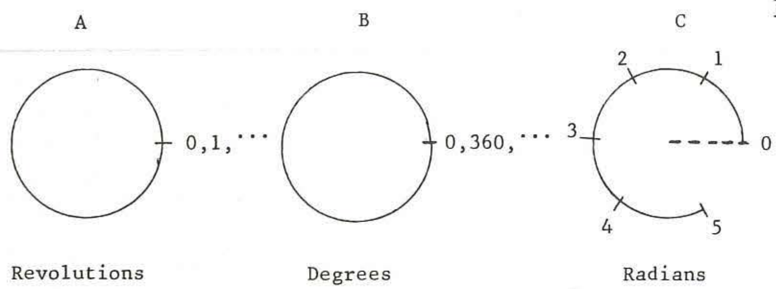


Figure 6.

This is strongly suggestive of traveling around and around in a circular path.

For a rapidly turning wheel or axle, it is convenient to assign the number one to a single complete turn or revolution. Let one revolution be divided into 360 equal parts, or degrees, as the ancient civilizations have taught us. Then many fractional parts of a turn are now measured with whole numbers: ½ turn is 180 degrees, ⅓ turn is 120 degrees, ¼ turn is 90 degrees, etc.

Another way to assign measure to circular arcs is shown in C of Figure 6. In a certain sense we permit the circle to decide its own measure. The size of a circle is fully determined by the choice of the length of its radius. Imagine a flexible tape on which the distance from zero to one is the same as the length of the radius of the circle, that is, the distance from the center of the circle to the circle itself. Begin at some point zero and wrap the tape around the circle. Then as in C of Figure 6 we have a picture of 5 as given in radian measure. This mental image of numbers is invaluable for many applications of mathematics in the field of calculus. This positive number 5 is measured in a counterclockwise direction; negative numbers are measured clockwise.

The association of numbers with ratios is very ancient, going back at least as far as the Greeks. Sir Isaac Newton (1769) considered the idea of ratio to be so basic that he used it in the definition of number: "By number we understand, not so much a multitude of unities, as the abstracted ratio of any quantity to another of the same kind, which we take for unity."

This way of thinking about numbers was given a concrete model by the Belgian educator, G. Cuisenaire, who introduced the colored rods which now bear his name. In Figure 7 if the white rod is assumed to have a measure of one, then 5 will be the measure of the yellow rod. The rods are unmarked, being identified only by color. This encourages a wide generalization: If any rod is assigned any positive number,

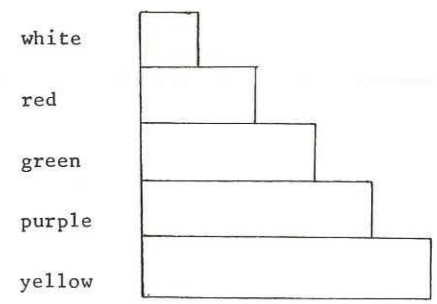


Figure 7.

then the ratio relations fix the unique number to assign to each of the other rods. For example, the purple rod is always twice as long as the red. If a number is assigned to either of these rods, then a number is fixed to assign to the other. Because of the three interpretations that can be given to x/y, these rods can be used to exemplify many properties of fractions and quotients, as well as ratios. Space does not allow discussion of their limitations, although the lack of a zero rod is evident.

Some non-linear models

The Cuisenaire rods vary in only one dimension of space, that of length. For two dimensions, with width or height as well as length, the square of unit sides and unit area is the fundamental unit. This is a difficult step for the learner in mathematics. Just a glance at Figure 8 is enough to reveal serious shocks to our intuitions. Certainly it is hardly evident that a square of area 4 is exactly twice the size of a square of area 2. Nor is it apparent that an area of 2 square units combined with an area of 3 square units should be equivalent to an area of 5 square units. A considerable amount and variety of "hands on" experience is a prerequisite before such relations can be made reasonable to our minds. Again, the difficulties with the ratios of areas are intensified when the ratios of volumes are considered. Here our intuitions are so strained that some might want to question the accuracy of the drawings for Figure 9.

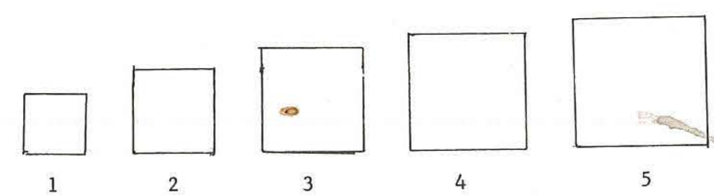


Figure 8.

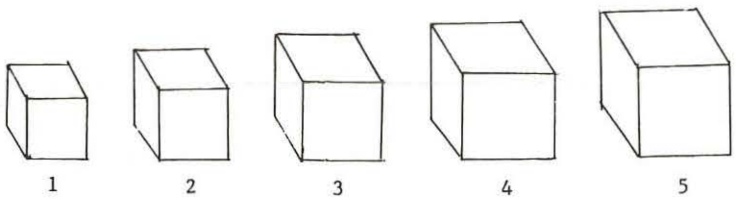


Figure 9.

Numbers as points

The symbols we call fractions—either the common variety such as ¾ or those called decimal fractions such as .75—are the numerals for the positive rational numbers. These numbers present conceptual difficulties far greater than those that arise when only counting numbers are considered. One of these mind stretchers is the loss of "nextness," which was an essential feature of the counting sequence in Figure 1.

What is the next larger fraction after ⅔? The answer is this: Such a number does not exist; it cannot even be imagined. True, ¾ is slightly larger, but 17/24 is larger than ⅔ or 16/24, yet 17/24 is also smaller than ¾, or 18/24. In fact, if x and y are two unequal rational numbers there is an infinite list of numbers that lie between them. For such a reason the system of rational numbers is said to be *dense*.

How can the mind be expected to visualize a dense set of numbers? The best answer we have is to adopt still another way of thinking about numbers, as suggested in Figure 10. On a line, extended in either direction as necessary, two distinct points are chosen. The number zero is assigned to one of these points, and the number one is assigned to the other. The line segment whose endpoints are zero and one then becomes the unit of length. The methods of geometry allow us to locate other points by adding, subtracting, multiplying, and dividing duplicates of this unit segment. Points determined in this way are rational points on the number line; each is associated with a unique rational number. The number five is now a point on this line.

Such a mental construct provides a model for a dense set of numbers, such as all the rational numbers between 1 and 4; we think of them as points on the line segment whose endpoints are 1 and 4. The points of a line do form a dense set. Between any two distinct points on a line there is another point, and even an infinite number of points. We have said that with each rational number there can be associated a unique

Figure 10.

point. But this does not mean that to every point can be assigned a rational number. Since Greek times it has been known that there is a number between 1 and 2, called a square root of 2 and written as $\sqrt{2}$, which is needed for measurement but that cannot be a rational number. Such numbers, and the points matched to them, are said to be irrational. Rational points are dense everywhere along the line, yet modern research has concluded that the set of points missed by the rationals—the irrationals—is also dense and in some sense there are even more of them than there are of the rationals. Thus we see how our search for a mental image for numbers has led to some very deep and imponderable properties. When we grant that for each counting number there is always a larger number, we are led to infinity in the large. When we reflect on the number line, we are confronted with infinity in the small.

For the counting sequence in Figure 1 every number has a unique successor. And every number, *except zero,* has a unique predecessor. If we allow every number to have a unique predecessor, the result is the sequence of integers. The notation for the integers is symmetrical, with zero as the center of symmetry. With each counting number different from zero there is paired its opposite; that is named by the same numeral, but by also including a minus sign as a tag to distinguish it from its partner.

The counting numbers provide answers for "How many?," and the rational numbers do this for "How much?" The extension to the negative numbers is useful for "Where?"

In Figure 11 the sequence of integers is used to extend the numbering of points for Figure 10; thus completing a figure that is called the real number line.

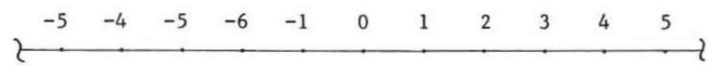


Figure 11.

Vectors for Numbers and Operators

There is still another visual image that we can use with great advantage for our example, the number 5. For each point on the real number line, except zero, we can associate a directed line segment, or vector. This vector will have its initial point at zero and its terminal point at the number by which it is named. The arrowhead at the terminal point gives the vector a sense of direction which is lacking for an undirected line segment. If the point zero is accepted as a limiting or degenerate vector, the vector model for each real number is then complete.

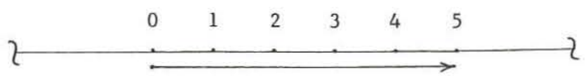


Figure 12. The vector should lie along the line, but is offset here for clarity.

Before presenting our final representation of number, we mention a caution to be learned from Skemp (1971) in his instructive discussion of mathematical symbolism and imagery. A visual symbol can convey a distinctive and even a dramatic message, but it can also be imprecise in this communication. Two persons looking at a mathematical symbolism or a visual model of it do not necessarily "see" the same thing. A simple and yet very fundamental example can be given.

There are several ways to think about even such a simple form as 3 + 2. We have first been taught to think of this in terms of a binary operation (an operation on two numbers); which can be emphasized by underlining, 3 + 2. Three is a first number and two is a second. The plus sign links these to form the composite symbol, 3 + 2. The arithmetic student may be encouraged to think of the plus sign as suggesting an operation (addition) yet to be done, to get the number 5 which will be called the sum of 3 and 2. Yet the algebra student must accept this addition as already accomplished by the writing of 3 + 2. This is a result of accepting 3 + 2 = 5 as a true statement because 3 + 2 and 5 are names for the *same* number. The change from 3 + 2 to 5 must be recognized as a change of form but not a change in amount. It is unfortunate that elementary texts commonly gloss over this conflict of meanings.

But the end is not yet for 3 + 2. This time we underline to suggest a unary operation meaning, 3 + 2. Three is still a first number, but the composite symbol + 2 now represents not a number, but rather a change. We think of + 2 as an operator, representing an increase of 2. The number 3 is the operand. When the operator + 2 is joined, by writing it at the right, the result is the transform, 3 + 2, which represents a second number.

Unary operations have possibilities for introducing a dynamic point of view into arithmetic which is yet to be recognized by texts and teachers. Curiously enough, some of the present practices already seem to follow the unary operation concept. For example, in presenting 3 + 2 = 5, a textbook picture may show a static model for the

number 3, such as three children standing in a group. But many such illustrations include a dynamic model, not for 2 but for + 2, as shown by two children at a distance but running to join the others. This leaves the number 3 + 2, or 5, to be imagined as the state after the two groups have become one.

Even the vertical display for suggested addition computations has a slight bias toward a unary operation.

3
+2 , suggests a 3 + 2, a unary operation

3
+2, would be a better binary sumbolism

If we return now to Figure 12, a fortunate circumstance can be observed. The vector model for the positive number, 5, can serve equally well as a mental image for + 5, that is, for an increase of 5. For a decrease of 5, as indicated by the operator − 5, the vector would have the same length as for + 5, but with the arrowhead moved to the other end. In summary, the length of the vector can give the amount of the change, while the two senses of direction along the line can differentiate between the two opposite kinds of change.

However, the single vector shown in Figure 12 is too limited in its portrayal of an increase of 5. The vector shown there is also the position vector for the number 5. As such, it is a bound vector, with its initial point necessarily at the origin.

But we want to think of increases as beginning at any chosen number (or point on the line). For this we need a free vector, that is, a vector free to move along a line but without changing its length or sense. We therefore enlarge our vector concept to include an equivalence class of vectors. (Two vectors are equivalent if they agree in length, direction, and sense.) As long as these conditions are met the vector remains equivalent even though it is translated to a new position.

The vectors for Figure 13 all represent increases of 5, even though they have been shifted to the left or to the right. Again we are to think of their acting along the line, even though they are here moved down for clarity.

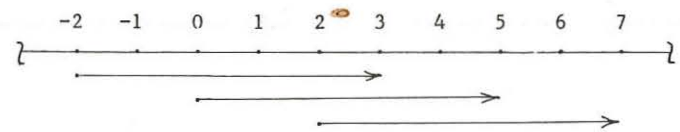


Figure 13.

As given here, our final interpretation for the real numbers will be to identify each with an equivalence class of vectors. For our example, the number 5, these will be the vectors of length 5 that are directed in a positive sense (as from zero to one). Space does not allow a demonstration of how this correspondence between numbers and vectors provides a foundation for the study of signed numbers.

Also omitted is the necessarily extensive discussion of the various binary and unary operations on numbers. We would find that there are distinctive advantages and disadvantages for each of these varied visual models as we consider such operations as addition, subtraction, multiplication, and division. Our purpose has been limited to suggestion of the rich field of investigation that exists in the visual imagery that can be associated with numbers.

Our society presents an ever increasing demand that mathematical competence be extended to a larger portion of its members. To make this possible we need to seek a better understanding (a better mental picture?) of number and its uses, and this properly begins with a study of its simplest ideas.

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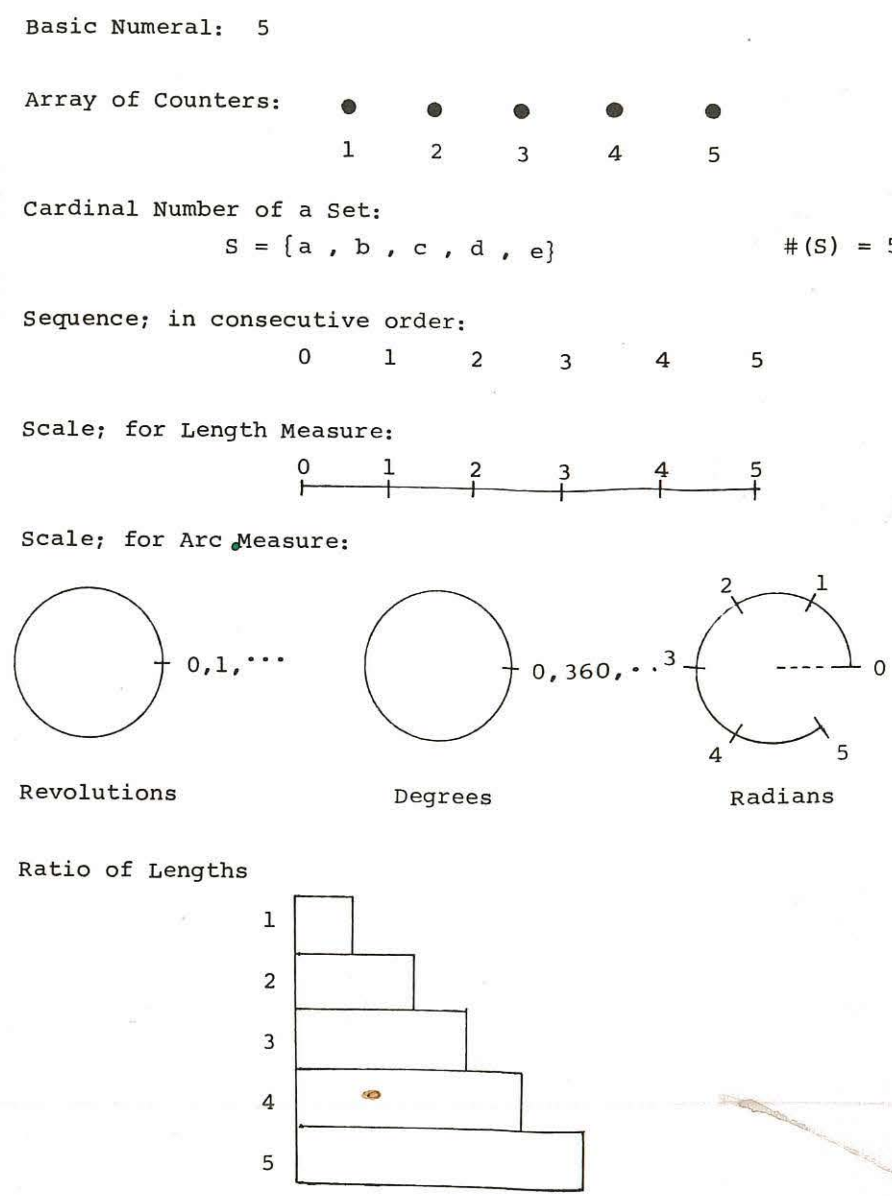
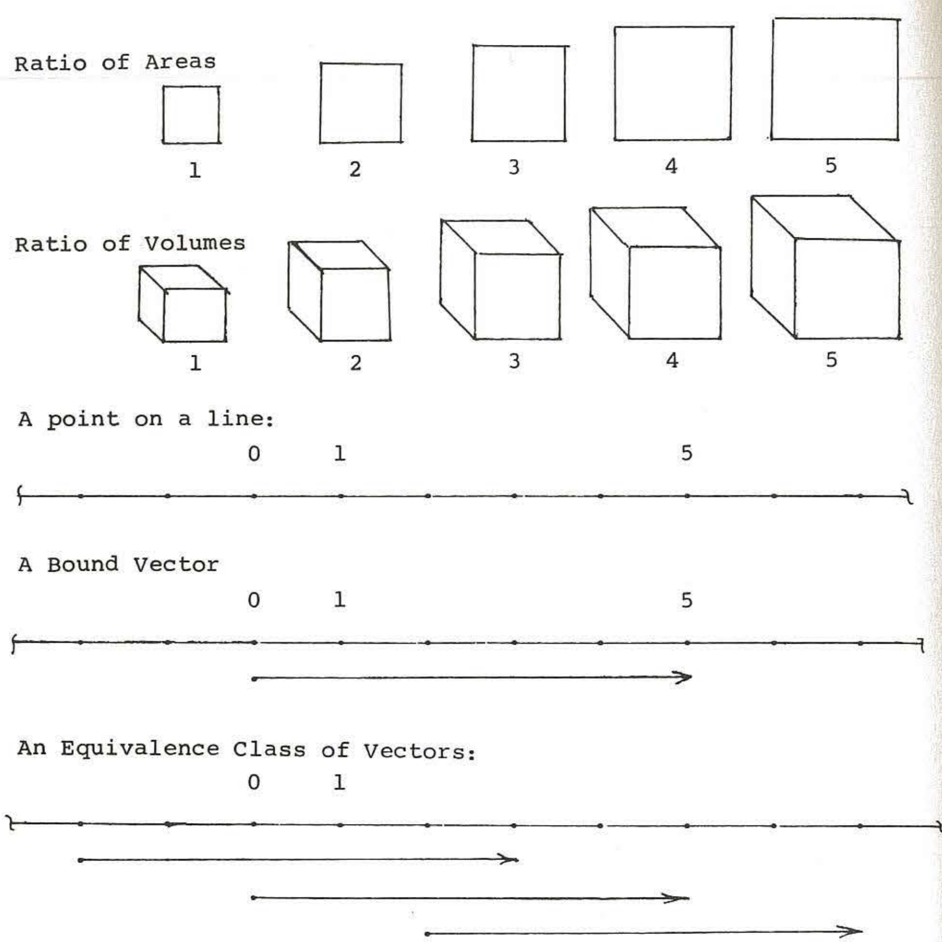
Representation of number





An equivalence class of vectors can also correspond to an increase of n, as represented by the operator + n.

Language Acquisition through Mathematical Symbolism

Francis Lowenthal

We noticed that the use of a non-verbal formalism can favour cognitive development (in the frame of the elementary school) in problem children as well as in normal children. An example is given to show how a formalism inspired by mathematics can be used to aid the development of the verbal language of 8- to 9-year-olds. We will then analyze the results and try to discover the cause of success we observed.

First we must specify which symbolic systems and which mathematical formalisms to use. In a previous paper (1980a) we stated, "We think that the main factor of cognitive development is *manipulation of representations*." In another paper (1980b) we claimed that any representation system which satisfies the six following criteria can be used: the system must be *non-ambiguous, simple and easy to handle, non-verbal* (to avoid conflicts with the developing verbal language); it must also be *supple enough* to enable the child to become conscious of what he knows but cannot verbally express; it seems essential that such a system should be *suggestive of a logic* and could be introduced and used *in the frame of games* (to enable us to use it easily with young children).

We wanted each of our systems to be suggestive of a logic; this is why we decided to choose representation systems used in mathematics. This requirement enabled us to represent our symbolic system in terms of a game. The rules of a game are explained and the children must collectively build a representation. This is the first stage of their work: *the synthesis*. They must then modify the representation and only respect technical constraints while doing so. They then reach the last stage: the analysis of the new representation and the collective discovery of the rules of the new game. Similar exercises can be invented for language acquisition.

What follows is a report of an actual lesson during which we asked the children "to tell a coherent story corresponding to a given representation." We will thus describe the adventures of a class of normal 8- to 9-year olds. The representation system we chose is that which is used in the new math (Papy 1968). Objects are represented by

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dots and relationships between objects by multicoloured arrows. Each dot represents exactly one object (which can have several names) and each colour represents exactly one relationship; 2 dots are associated to 2 different objects and 2 colours to 2 differing relationships. The children suggested the starting diagram (Figure 1). They decided to use only two kinds of "arrow-relationships": red and green ones. (For technical reasons we will represent red by a discontinuous line and green by a continuous line.)

First stage

Ronald produces the diagram shown in Figure 1, but does not say anything. Rudy asks immediately: "Does one split Magali into two?" but Ronald does not answer. Fabrice notices: "The dot below has no name," and Rudy tries to explain: "The green arrow says, 'to go to the park,' so Magali goes to Nicolas' and Nicolas goes and sleeps in the park."

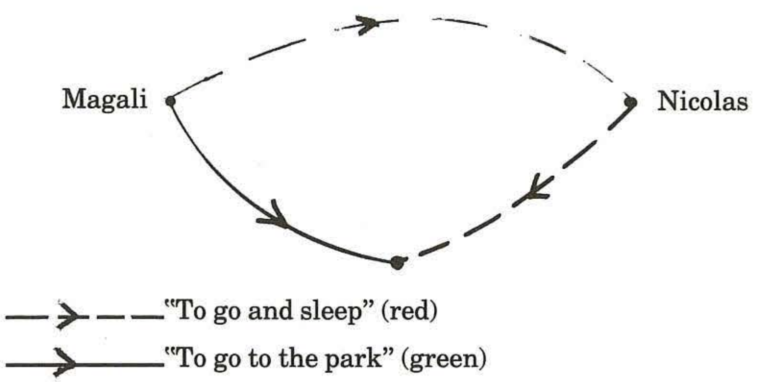

Figure 1.

Second stage

Isabelle suggests calling the third dot "Marie" and the whole class accepts this. Rudy, who is still thinking in terms of games, says: "One game, it will be the park; the other one, it will be "to sleep." We should add more arrows." (He probably assumes that there are two "games" for Magali).

Fabrice asks: "Does Magali go to the park at Marie's?" He adds: "Magali goes and sleeps at Nicolas'. Nicolas goes and sleeps at Marie's. Magali goes to the park at Marie's." Pascal corrects him: "...with Marie."

The teacher interferes then and asks: "Fabrice made a mistake. Why?" Isabelle suggests: "Magali goes to the park at Marie's." "At Marie's?" asks the teacher. The children do not seem to find improper the word "at." The teacher insists then upon the relationship "to go and sleep." Catherine re-reads the picture: "Magali goes and sleeps at Nicolas'." Fabrice suggests: "Magali goes and sleeps Nicolas" but Catherine proposes: "'Magali, go and sleep!' says Nicolas"; and Pascal: "Magali goes and sleeps with Nicolas."

"'At' or 'with'" asks the teacher, while Rudy wonders: "May we change the arrows?" Fabrice wants to add a little word to "to go and sleep" and obtains: "to go and sleep *at* [...'s house]." The class thus obtains a text which is read by Silvie: "Magali goes and sleeps at Nicolas'. Nicolas goes and sleeps at Marie's."

"The green arrow annoys me," announces Rudy. Bertrand notices: "It gives: Magali goes to the park Marie." "To go to the park *of* Marie" says Rudy, but this is rejected by the rest of the class, while Catherine suggests adding "with" at the dot called "Marie." The text becomes then: "Magali goes and sleeps at Nicolas'. Nicolas goes and sleeps at Marie's. Magali goes to the park with Marie."

Third stage

Pascal notices that "If Magali goes and sleeps, she cannot go into the park." Fabrice puts both actions in a time perspective: "Magali goes and sleeps at Nicolas'. Tomorrow Nicolas will go and sleep at Marie's. The day after tomorrow Magali will go to the park with Marie."

The teacher, thinking of the symmetry implied by the word "with," asks: "There is a problem in this story. Which one?" He has the impression that the pupils feel that there is a qualitative difference between the two actions of Magali, but that they cannot express verbally and correctly the idea.

"If Magali goes to the park with Marie..." starts the teacher, and Fabrice continues: "Then Marie goes to the park with Magali." The pupils then suggest adding an arrow and obtain the diagram shown in Figure 2. Catherine notices: "Both are going," and the teacher adds: "To go somewhere with somebody; the persons are together."

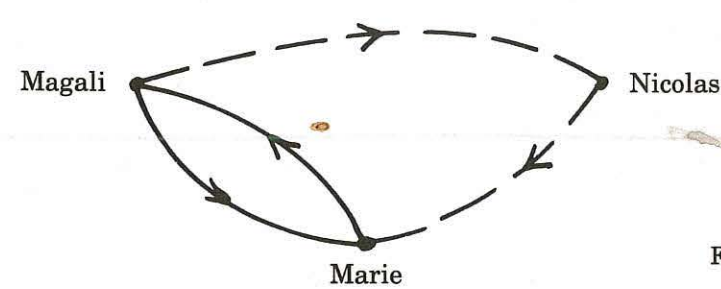

Figure 2.

Sylvie finally tells the complete story: "Magali goes and sleeps at Nicolas'. Nicolas goes and sleeps at Marie's. Magali goes to the park with Marie. Marie goes to the park with Magali." The whole class gives these data the shape of a story: "Magali goes and sleeps at Nicolas'. The next day Nicolas will go and sleep at Marie's. The day after tomorrow, Magali and Marie will go together to the park." Catherine uses the time variable to create another story, using the same basic data: "Yesterday Magali went and slept at Nicolas'. Yesterday Nicolas went and slept at Marie's. To-day, Magali and Marie will go to the park together."

During this lesson the children successfully and graphically produced a situation. Starting from this situation, they created a story. Their teacher can now use the story *created by the children themselves* to introduce, in the frame of the language course, exercises about conjugation, about personal or relative pronouns, or even subordinates.

Analysis of this example

At the first stage the children create a simple situation: there are objects and relationships. We are at the object-language level, not at the metalanguage level. There is not really a story. At the second stage we notice that the diagram suggests (to the children) complements which are required from a logical point of view (naming of all the dots, correct statement of all the relationships). There are already sentences, but not yet a story.

During the third stage the children correct apparent contradictions, thanks to the explicit introduction of variables which remain implicit in an ordinary conversation. This concerns the properties of certain words (e. g., symmetry for "with") but mainly the introduction of the time variable. The children are, nearly constantly, the initiators of the action, not the teacher. The children choose freely the conventions they want to use. The classroom dynamics plays an important role: one child proposes an idea, the whole class criticizes it, and after discussion all accept it — or reject it.

Discussion

One diagram is used as starter. During their discussion the children modify many things. They first try to adapt the story to the diagram, then the diagram to the new version of the story. It seems important to notice this back-and-forth process during which, in this case, time is introduced. More generally comments about the representation and

assessments of the value of one diagram compared to another appear in the children's speech.

Object-language is that part of language concerning descriptions of objects, or relations between actual objects, while metalanguage is that part of language concerning what is said about these descriptions or relations. "My pencil is broken" belongs to object-language, but "'My pencil is broken' is a correct sentence" belongs to metalanguage. When we use the technique described in this paper, we pass easily, especially in the way time is introduced, to the level of metalanguage and the children eventually create a coherent story. We think that our symbolic representations are useful, mostly because they enable the children to distinguish clearly between object-language (associated to the representations) and metalanguage (what is said about the representations). We think that this concrete distinction between object-language and metalanguage might be the factor which favours the children's cognitive development. A young child is able to use representations, but not always to state or to notice that two different representations can be used for the same object, for the same story. This is a problem of non-identity or non-conservation. But this is no longer true if we ask the child to compare concrete representations and to *say what he notices* while doing so: the child will then rapidly learn what Piaget said this young child cannot learn.

Remarks

The representation system we used enabled the children to create a diagram. This diagram was only used as starter for the narrative process. The story the children created is wider than the diagram's frame. Moreover, although it is true that the construction of the story is based upon the diagram (the basic elements of the story are suggested by the diagram), it is nevertheless wrong to believe that the diagram is used to communicate a complete message. There are conventions established by the pupils, many things are implicit and are not mentioned; other problems are never solved. It is true that Magali goes and sleep at *and with* Nicolas (story told by Pascal and Sylvie)? Or does Nicolas go away and leave Magali alone (story told by Catherine)?

Mistakes to avoid

The pupils should build their story by themselves; the teacher should only guide them when needed, as little as possible. Our technique should certainly not be used too formally; it would block the pupils' activity. We must accept, for instance, the quasi-identification of "Nicolas" and "at Nicolas' house," accept that the pupils change the name of the third dot and call it "with Marie" instead of keeping "Marie," and explicitly add the "with" to the green relationship. We may never forget that it is difficult to tell a story corresponding to the starting diagram. The teacher must be supple and let the children modify the diagram when they want to do it and exactly as they want to do it. They must have the possibility to adapt the diagram to the story they wish to tell, and the story to the diagram they want to keep, in such a way that they slowly reach a solution which satisfies them. If they succeed in building a coherent story — but a story which does not even look like the starter — the teacher must be able to accept it: the terminology, the convention, the game's control belong to the children.

One must, at all cost, avoid dogmatic use of the technique, for dogmatism kills the children's freedom of expression. We must use representation systems which, thanks to inner technical constraints, suggest to the child the use of a logic which the teacher has hidden in it.

Conclusion

A non-verbal auxiliary formalism can serve as guide to the child's thought. If this formalism, or representation system, is used in a non-dogmatic way, it enables the children to build a coherent story through successive adaptations that they suggest. This story can be graphically represented by the proposed formalism. In this case one should use the definitions formulated by some children and accepted by the whole class. Such a formalism is also useful because the teacher, when choosing the symbols and imposing upon them the technical constraints, can hide a logic in the system. The teacher can thus choose a logic which the children will use nearly spontaneously. Moreover, such a formalism enables the teacher to visualize the difference between object-language and metalanguage.

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Communicating Mathematics: Surface Structures and Deep Structures

Richard R. Skemp

A distinction is made between the surface structures (syntax) of mathematical symbol-systems and the deep structures (semantics) of mathematical schemas. The meaning of a mathematical communication lies in the deep structures — the mathematical ideas themselves, and their relationships. But this meaning can only be transmitted and received indirectly, via the surface structures; correspondence between deep and surface structures is only partial. Some resulting problems of communicating mathematics are discussed, and some remedies suggested.

The power of mathematics in enabling us to understand, predict, and sometimes to control events in the physical world lies in its conceptual structures — in everyday language, its organised networks of ideas. These ideas are purely mental objects: invisible, inaudible, and not easily accessible even to their possessor. Before we can communicate them, ideas must become attached to symbols. These have a dual status. Symbols are mental objects, about which and with which we can think. But they can also be physical objects — marks on paper, sounds — which can be seen or heard. These serve both as labels and as handles for communicating the concepts with which they are associated. Symbols are an interface between the inner world of our thoughts, and the outer, physical world.

These symbols do not exist in isolation from each other. They have an organisation of their own, by virtue of which they become more than a set of separate symbols. They form a symbol system. A symbol system consists of

a set of symbols	corresponding to	a set of concepts

together with

a set of relations between the symbols	corresponding to	a set of relations between the concepts.

What we are trying to communicate are the conceptual structures. *How* we communicate these, or try to, is by writing or speaking symbols. The first are what is most important. These form the *deep structures* of mathematics. But only the second can be transmitted and

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received. These form the *surface structures.* Even within our minds the surface structures are much more accessible, as the term implies. And to other people they are the only ones which are accessible at all. But the surface structures and the deep structures do not necessarily correspond, and this causes problems.

Here are some examples to illustrate the differences between a surface structure and a deep structure.

I feel like a wet rag → Same surface structure, different deep structure ← I feel like a glass of beer

I feel like a glass of beer → Same surface structure, same deep structure ← I feel like a cup of tea

I feel like a cup of tea → Different surface structure, same deep structure ← Shall I put the kettle on?

What has this to do with mathematics? At a surface level wet rags and cups of tea would seem to have little connection with mathematics. But at a deeper level, this distinction between surface structures and deep structures, and the relations between them, is of great importance when we start to think about the problems of *communicating* mathematics.

For convenience let us shorten these terms to S for surface structure, D for deep structure. S is the level at which we write, talk, and even do some of our thinking. The trouble is that the structure of S may or may not correspond well with the structure of D. And to the extent that it does not, S is inhibiting D as well as supporting it.

Let us look at some mathematical examples. We remember that a symbol system consists of:

(i) a set of symbols, e.g. 1 2 3 ...
$\frac{1}{2}$ $\frac{3}{4}$ ...
a b c ...

(ii) one or more relations on those symbols, e.g. order on paper (left/right, below/above); order in time, as spoken.

But since the essential nature of a symbol is that it represents something else — in this case a mathematical concept — we must add

(iii) such that these relations between the symbols represent, in some way, relations between the concepts.

So we must now examine what ways these are, in mathematics. Here is a simple example. (Note that 'numeral' refers to a symbol, 'number' refers to a mathematical concept.)

Symbols	*Concepts*
(i) 1 2 3 ... (numerals in this order)	the natural numbers
Relations between symbols	*Relations between concepts*
(ii) is to the left of (on paper) before in time (spoken)	is less than

This is a very good correspondence. It is of a kind which mathematicians call an isomorphism. Place value provides another well known example of a symbol system.

Symbols	*Concepts*
(i) 1 2 3 ... (numerals)	natural numbers
Relations between symbols	*Relations between concepts*
(ii) numeral_1 is one place left of numeral_2.	number_1 is ten times number_2.

By itself this is also a very clear correspondence. But taken with the earlier example, we find that we now have the same relationship between symbols, *is immediately to the left of,* symbolising two different relations between the corresponding concepts: *is one less than* and *is ten times greater than.* We might take care of this at the cost of changing the symbols, or introducing new ones; e.g., commas between numerals in the first example. But what about these?

23 $2\frac{1}{2}$ 2a

These can all occur in the same mathematical utterance. And this is not just carelessness in choice of symbol systems; it is inescapable, because the available relations on paper or in speech are quite few: left/right, up/down, two dimensional arrays (e.g., matrices); big and small (e.g., 7, r) What we can devise for the surface structure of our symbol system is inevitably much more limited than the enormous number and variety of relations between the mathematical concepts, which we are trying to represent by the symbol system.

Looking more closely at place value, we find in it further subtleties. Consider symbol: 5 7 2. At the S level we have three numerals in a simple order relationship. But at the D level it represents

(i) three numbers, corresponding to 5 7 2

↕ ↕ ↕

(ii) three powers of ten: 10^2 10^1 10^0
These correspond to the three locations of the numerals, in order from right to left.

(iii) three operations of multiplication: the number 5 multiplied by the number 10^2 (= 100), the number 7 multiplied by the number 10^1 (= 10), the number 2 multiplied by the number 10^0 (= 1).

(iv) addition of these three products (5 hundreds, seven tens, two).

Of these four at D level, only the first is explicitly represented at S level by the numeral 572. The second is implied by the spatial relationships, not by any visible mark on the paper. And the third and fourth have no symbolic counterpart at all: they have to be deduced from the fact that the numeral has more than one digit.

Once one begins this kind of analysis, it becomes evident there is a huge and almost unexplored field—enough for several doctoral theses. For our present purposes, it is enough if we can agree that the surface structure (of the symbol system) and the deep structure (of the mathematical concepts) can at best correspond reasonably well, in limited areas, and for the most part correspond rather badly.

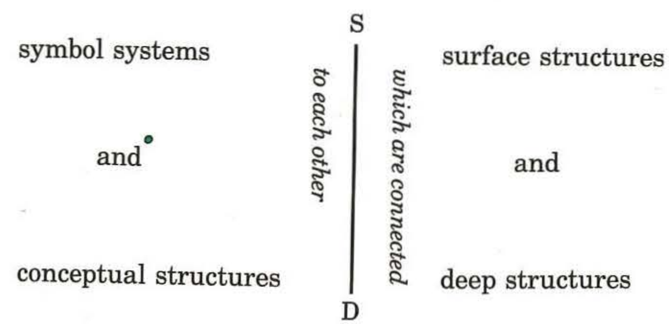
To help our thinking further in this difficult area, I would like to introduce two further ideas. The first comes from my new model of intelligence (Skemp 1979) and does not require any other parts of the theory. It is based on the well-known phenomenon of resonance. "The starting point is to suppose that conceptualised memories are stored within tuned structures, which, when caused to vibrate, give rise to complex wave patterns.... Sensory input which matches one of these wave patterns resonates with the corresponding tuned structure, or possibly several structures together, and thereby sets up the particular wave pattern of a certain concept." (page 134)

It is convenient at this stage to introduce the term *schema,* which is simply a shorter way of referring to a conceptual structure. A schema (i.e., a conceptual structure stored in memory) thus corresponds in this model to a particular tuned structure. We all have many of these tuned structures corresponding to our many available schemas, and sensory input is interpreted in terms of whichever one of these resonates with what is coming in. What is more, different structures may be thus activated by the same input in different people, and at different times in the same person. Different interpretations will then result. For example, the word

'field' will have quite different meanings according as it evokes resonances corresponding to the schemas in advanced mathematics, electromagnetism, cricket, agriculture, or general scholarship.

The second idea is due to Tall (1977) who has suggested that a schema can act as an attractor for incoming information. He took the idea from the mathematical theory of dynamic systems; but if we now combine it with the resonance model, we can offer an explanation of how this attraction might take place. Sensory input will be structured, interpreted, and understood in terms of which ever resonant structure it activates. In some case, more than one resonant structure may be activated simultaneously, and we can turn our attention at will to one or the other. In others, one schema captures all the input. (This 'capture effect' is well known to radio engineers, who have put it to good use.)

So we may now synthesise the following ideas.



Note that in the above diagram each point represents not a single concept but a schema, in the same way as a dot on an airline map can represent a whole city—London, Atlanta, Rome.

How can this theoretical model help our thinking, and what are the practical consequences? All communication, written or oral, is necessarily into the symbol system at S. *To be understood mathematically, it must be attracted to D.* This requires that D is a stronger attractor than S. If it is not, *S will capture the input,* or most of it.

One of the advantages of a good model is that it points up some questions we should ask next. The first is clearly: What are the conditions for D to be a strong attractor? Another is: can D capture the input instead of S? If so what happens?

I will take the second first, briefly. If this were to happen, I think it would mean that all the mathematical activity was confined to a deep conceptual level, and was not 'escaping' to a symbolic level at all. This may not happen completely, but some of the high-powered mathematicians who taught me at university suggest only very limited escape to S!

Returning to the first question: what are the conditions for D to be a strong attractor? S has a built in advantage: all communicated input has to go there first. And for D there is a point of no return. In the years' long learning process, if the deep conceptual structures are not formed early on, they can never develop as attractors. For too many children, D is effectively not there. And if the D structure is absent or weak, all input will be assimilated to S: the effort to find some kind of structure is strong. So S will build up at the expense of D.

But this guarantees problems, in view of the lack of internal consistency of S. This reveals a built-in advantage of D, that it *is* internally consistent. Of all subjects, mathematics is one of the most internally consistent and coherent. So if it gets well established, input to S will evoke more extensive and meaningful resonances in D than in S, and D will attract much of the input.

Doing mathematics involves the manipulation of certain mental objects, namely mathematical concepts, using symbols as combined concepts and labels. But for many children (and adults) these objects are not there. So they learn to manipulate substitute objects: empty symbols, handles without anything attached, labels without contents. This in the long run is much more difficult to do, though unfortunately in the short run it may be easier to learn. The manipulation of mathematical concepts is helped by the nature of the concepts and schemas themselves, which give a feeling of intrinsic rightness or wrongness. This arises partly from the concepts themselves, whose individual properties contribute to how we use them and fit them together. More strongly, it comes from the schemas, which determine what are permissible and non-permissible mental actions within a given mathematical context.

The problems which so many have with mathematical symbols thus arise partly from the laconic, condensed, and often implicit nature of the symbols themselves; but largely also from the absence or weakness of the deep mathematical schemas which give the symbols their meaning. Like a referred pain, the location of the trouble is not where it is experienced. The remedy likewise lies mainly elsewhere, namely in the building up of the conceptual structures.

How can we help learners to do this? This is too large a question for a single paper, but here are some suggestions as starting points.

(i) Particularly in their early years we can give children as many physical embodiments as possible of the mathematical concepts which we want to help them to construct. As examples of units, tens, and hundreds, we can use single milk straws, bundles of ten of these, and bundles of ten bundles of ten. These correspond much more closely to the relevant mathematical concepts than do the associated symbols, and so the visual input will be attracted more strongly to the relevant parts of D than to S. In such cases, moreover, the input goes first to D, then to S, since the children are first presented with the physical embodiments of the concepts, and thereafter are asked to connect these with appropriate symbols.

(ii) By careful analysis of the mathematical structure to be acquired, we can sequence the presentation of new material in such a way that it can always be assimilated to a conceptual structure, and not just memorised in terms of symbolic manipulations. Many existing texts show no evidence that this has been done. (See Skemp 1971, Chapter 2.)

(iii) Again in these important early years, it helps children if we stay longer with spoken language. The connection between thought and spoken words are initially much stronger than those between thoughts and written words or symbols. Spoken words are also much quicker and easier to produce. So in the early years of learning mathematics, we may need to resist pressures for children to have 'something to show' in the form of pages of written work.

(iv) It is often helpful to use informal, transitional notations as bridges to the formal, highly condensed notations of traditional mathematics. By allowing children to express their thoughts in their own ways to begin with, we are using symbols which are already well attached to their associated concepts. These ways of expression may often be lengthy, unclear, and differ between individuals. By experience of these disadvantages, and by discussion, children may gradually be led to the use of established mathematical symbolism in such a way that they experience its convenience and power for communicating and manipulating mathematical ideas.

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Mathematical Symbolism

Derek Woodrow

One of the essential distinguishing features of mathematics is its eventual dependence upon symbols and symbolic expression. Few attempts to determine those processes, activities, or contents which uniquely identify mathematics have succeeded. It is indeed questionable whether human knowledge can be classified into such self-contained categories. The many diverse activities of mathematicians do, however, have symbolic expression as their common feature, and the extent to which modern disciplines depend upon mathematics could be measured by their growing reliance on symbols. It is reasonable to surmise that much of the difficulty experienced by children in mathematics, and the lack of popularity of physical as opposed to biological sciences in higher education, could be traced to the problem of symbolisation. It will be interesting to watch the effect on, say, geography as the school syllabuses move towards mathematical as opposed to descriptive aspects. There is surprisingly little apparent research into the use and learning of symbols, except for the many investigations into both the problem of how children learn to read and adult perceptual experiences with words (e.g., Coltheart 1972). There is, however, a real distinction between the use of symbols as a verbal language (spoken or written) and the use of symbols in the mathematical sense. It will indeed be suggested below that one activity interferes with the other.

In normal reading activity the written word contains very many redundancies. There is clear experimental evidence that not only are many of the words used unnecessary and the number of letters per word quite extravagant but the letter symbols themselves are only partially scanned in many reading techniques. The reader only notices, say, the bottom of the letter and the relationship between the symbols is sufficient to determine them completely. Try reading the following doggerel:

> THR NC WS YNG MN, WHS FC WS GRN?
> T WS TR THT LL WH SW HM FND HM TH STRNGST THNG
> THD SN

The relationships between verbal symbols can also be seen in the way in which adults react and remember random letters. A collection of letters such as POSTIC is much more easily read and remem-

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bered than XZBQT which proves much more difficult because it does not resemble the normal letter associations used in the English language.

The redundancy which is normal in language is not usually present in mathematical symbolism at school level. Statements such as:

$$3 + 4 + 10 + 3 \cdot 2 = 20 \cdot 2$$
$$A \cap B' = \emptyset$$
$$(3,4) + (4,5) = (7,9)$$
$$4x^2 + 3x + 2 = 0$$

contains little redundancy, although the last example *with experience* can be seen to have a recognisable form in which one might only need to know the coefficients 4, 3, and 2. Even in this case, however, the relevant distinctive information is contained *inside* the symbolisation which must therefore be read rather than just seen. Yet another complication in mathematical symbolism is the phenomenon of temporary redundancy in which a whole group of symbols are at one stage carried without reading, only to need detailed reading later. For example:

$$(12x^2 - 2)^2 - 2(12x^2 - 2) + 1 = 0 \qquad 12x^2 = 3$$
$$[(12x^2 - 2) - 1]^2 = 0 \qquad x^2 = \tfrac{1}{4}$$
$$12x^2 - 2 = 1 \qquad x = \pm\tfrac{1}{2}$$

This becomes more apparent in the later stages of learning mathematics, and this variation in the degree of redundancy causes many problems for college and university students.

Another distinction between the use of words and mathematical symbols is the independence of one symbol from the preceeding and succeeding symbols. The anecdote is related of the three-year-old who was arithmetically very advanced in that the addition of three digit numbers presented little difficulty. His parents expressed some concern that he had no interest in reading; reputedly because letters behave irrationally, in the sense that whilst any sequence of digits make a sensible number a random sequence of letters do not make a word. In reading, the individual symbols do not themselves contain any meaning, whereas in mathematics, with a few exceptions such as d/dy or ( ), the meaning of the individual symbols is vital.

Even more disturbing to the learner is the interrelationship of mathematical symbols where not only does each symbol have its own distinctive meaning, but this meaning is affected by its neighbouring symbols. Consider, for example, the schema attached to the symbol 2 in 212, ½, $\sqrt{2}$, $f(2)$, a_2, a^2, $\mathbb{R}^2$, 2'o-clock, 1001_2, (2,3), etc. In each case there are subtle changes in a basic schema which originally starts as a fairly low-level concept in 2 as used in the infant natural number

sequence but becomes a higher and higher level schema as mathematics progresses.

The essential concentration in school curricula on literacy tends to produce, therefore, a reading technique which to some extent interferes with the technique required in reading mathematical symbols. If one accepts this proposition, then two implications arise: we must adapt mathematical symbolism for the learner, and we must follow a careful and structured plan to teach the pupil how to read mathematics.

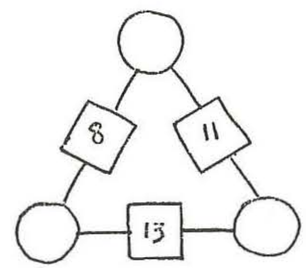

Figure 1. The arithmogon.

Signs

One of the usual ways of adapting mathematical symbols to the schema used by pupils is the use of signs such as boxes instead of symbols. Many books now in use make extensive use of boxes from a very early stage, frequently asking questions such as 3 + □ = 7. Another interesting example is the arithmogon (McIntosh & Quadling 1975). In the arithmogon (Figure 1) the numbers which belong in the square boxes are the result of adding the numbers in its adjacent circle boxes. Many points of interest arise from the investigation of what numbers should be in the circles for given numbers in the squares. What is relevant to the present argument is the difference in schema attached to this problem compared to its presentation in the usual mathematical notation. Whilst many primary children could tackle the sign statement of the problem, it is doubtful if many early secondary school pupils would be able to manage the symbolic statement.

$$\begin{cases} x+y=8 \\ x+z=11 \\ z+y=13 \end{cases}$$

Comparison of the two expressions 3 + □ = 5 and 3 + x = 5 illustrates some of this difference between signs and symbols. The first uses the sign □ to replace the missing number and the second uses x

in apparently the same way. The second expression carries with it, however, a much more abstract statement which says 'this particular example belongs to a whole class of things which can be dealt with in such-and-such a way.' In solving the first problem it really is the number which should be in the box which is the relevant factor, in the second it is the process of obtaining whatever number turns up which is relevant. Whilst this appears to be a post-operandi argument and to have import only at later stages in the learning of mathematics, experience points to the operation of such distinctions at an almost instinctive level. Children who cannot be at all aware of this distinction from experience react so differently to the use of a sign $\square$ than a symbol x.

Rather surprisingly this distinction in the order of concept involved is echoed in adult perception. Coltheart (1972) reports an experiment in which observers are presented with a 3×4 matrix of letters or shapes. After the display has been removed the observer is asked to remember a particular subset of the display chosen on the basis of position, colour, shape, or size. In the particular problem investigated this showed the existence of a short term memory of greater detail and scope than normal recall. What was rather surprising was that this short term intensive memory apparently failed to operate as effectively when the display was a mixture of letters and digits and this distinction was used as a discriminant. This would suggest that the ability to distinguish between letters and digits is in some respect different from discrimination in position, size, shape, or colour. This might indicate, incidently, another of the great advantages of arabic place-value notation based upon position rather than earlier hieroglyphic representations which depend upon a higher level of symbol discrimination. It would be interesting to repeat the recall experiments with young children to investigate if there is any particular age at which the distinction between signs and symbols, as defined here, becomes relevant. It seems very likely that the use made in mathematics of letters for numbers is probably neither accidental nor irrelevant.

It is clear that adults do not, indeed, experience much difficulty in handling signs in normal everyday life. There has always been an immediacy and ease in the use of signs for religious, political, and social reasons. Mere reference to scarab-beetles, fish, crosses, eagles, hammers and sickles, white feathers, tudor roses, fleur-de-lyse, and so on, produce immediate images and attract schemas from our memories which are full of vividness. Freud, and modern advertisers, have made this fully conscious. Traffic signs, laundry signs, and the markings on

electronic equipment illustrate the steady growth in the use of signs in modern life. The contrast of these signs with mathematical symbols illustrates the distinctive features of a sign, which is essentially a low level naming concept which identifies a single, identifiable, non-adaptable idea. Symbols, on the other hand, are identified with high-level schemas rather than concepts, and as such are more responsive to adaptations and multiple relationship. Three different types of symbolisation have therefore been identified:

Language symbols.Contain high redundancy, great interdependence, and no individual meaning.

Signs.Contain little redundancy, not interdependent and unaffected by neighbouring signs, represent single (naming) concepts.

Symbols.Contain little redundancy, interdependent and adaptable to neighbouring symbolisms, related to schema.

The Functioning of Symbols

Skemp (1971) suggests ten different ways in which symbols are used: i Communication, ii Recording, iii Forming new concepts, iv Aiding multiple classification, v Explanation, vi Aiding reflective mental activity, vii Exhibiting structure, viii Automating routine manipulations, ix Recovering information, and x Producing creative mental activity. Not all of these are, of course, independent and more than one mode of functioning is often at play at the same time. In Skemp's clear descriptions of these roles for symbolisation certain underlying problems and ideas can be seen. At a high level of mathematics there is a clear contradiction between two characteristics of symbolic representation; the condensation which symbols achieve contrasts with their use as a precise language. Both these aspects relate to the early learning of mathematics in which symbols are used to name concepts and schemas, and yet in different contexts we change and adapt these schemas to meet different needs, without always changing the symbol. (Perhaps we need vari-focal symbols to complement the idea of vari-focal concepts presented in Skemp 1979.)

Symbols as Names

Skemp comments 'It is largely by the use of symbols that we achieve voluntary control over our thoughts,' and the ability to name a thing has always conveyed controlling power in both Greek and Nordic mythology. In answer to 'What is the largest number,' the word 'infinity' settles all discussion, and the fact that the solutions to $x^2 - x + 1 = 0$ are complex satisfies most enquiries even though the hearer may have no clearly defined meaning for the words. Mathematics is usually concerned with higher order concepts for which the defining examples are other concepts, and these can only be expressed in verbal or symbolic form. Just as the young child must have the certainty of conservation of his physical observations before being able to operate with them, so the student of mathematics must be assured of the certainty of the lower level concepts before he can build with them. One of the major roles of symbols lies in communicating and holding these concepts with others, or with oneself in internal reflection and argument. In identifying three types of listener — the 'don't knows,' the 'want to know more,' and the 'critics' — Skemp illustrates the different contexts in which schema, and hence symbols, undergo subtle changes. These range from naming a general target area in which the concept is allowed to be fuzzy but the direction of clarification is hopefully indicated, through periods in which some concepts are clarified whilst others are left vague, until in the critical stage every symbol has its own specific and precisely defined meaning. The student not only passes through these stages in turn, but passes through them more than once as concepts are continually redefined. This is not only true of high level concepts such as integration, but even early in the secondary school level the uses of π and $\sqrt{2}$ illustrate the variety of conceptual contexts. Similarly the continual redefinition of multiplication has led to the introduction of the idea of group properties in an attempt to establish a conserved concept which is unvarying enough to be built upon. In the same way the idea of function compared to relation reflects a need to distinguish between two different uses of variables which otherwise cause a disturbing vagueness.

If communication is to be meaningful, it is clear that the symbol used to signal a schema in one person must signal the same schema in his correspondent. One of the problems in the use of symbols by pupils is that the teacher has frequently condensed his early use of multiple concepts and symbols into a single one. Thus, for example, the development of the concept of subtraction involves a variety of different lower level concepts such as 'take away,' 'how much bigger,' 'what is the difference between,' out of which is generated an underlying idea. Until the child has developed this underlying concept the use of the same symbol for different concepts can cause problems, and it is important that the symbolism should mirror the different activities. On occasions it is therefore necessary to use two or three symbols (or rather signs) in the early stages. (The reader is invited to describe the activities symbolised in the following list.)

$5-3=2$
$(5,3)\Rightarrow 2$
$5+\square=2$ $\longrightarrow$ $5-3=2$
$5\xrightarrow{-3}2$
$5+{}^{-}3=2$

The same tendency for symbols to outreach the attained concept can be seen in the use of ${}^{-}1$ and ${}^{+}3$ for the directed numbers, and many teachers have encountered difficulty which pupils have with notations such as $(\ )^{-1}$. This really denotes a multiplicative inverse, but in situations such as $\sin^{-1}x$ the connection with any multiplication situation is far from obvious. Even the equivalence of $\frac{3}{7}$ and $3 \div 7$ is not at all easy to establish.

The Introduction of Symbols

There is an apparent confusion in the work of both Skemp and Dienes on the introduction of the name of a concept (its symbol) early in the learning process. To quote from Dienes (1964): 'The most likely reason for the general ossification of mathematics in children's minds at an early stage is the rash use of symbols, i.e., the introduction and manipulation of symbols before adequate experience has been enjoyed of that which is symbolised' and 'the apprehension of structures and the symbolisation process are not altogether distinct, and in fact there is reason to believe that each acts as a stimulus on the other.'

Similarly Skemp (1971): 'Making an idea conscious seems to be closely connected with associating it with a symbol' and 'Concepts of a higher order than those which a person already has cannot be communicated to him by definition, but only by arranging for him to encounter a suitable collection of examples.'

Both writers are really talking about symbols as representing structures, (central unifying ideas, schemas) as compared to signs representing low level concepts. Skemp makes the point that there is a distinction between reflecting on content and reflecting on form which is relevant in this context, since the level of content is that of naming concepts. This distinction clearly relates to Piaget's distinction between concrete operational and formal operational thinking. The usually suggested ages for maturing from one mode to the other (in general between about 12 and 16 years of age) would indicate a need to persist with less flexible signs related to content rather than symbols related to form. The timing of this change from the particular to the abstract is implicit in the good teacher, but there has been little research to make it explicit and therefore more transferrable. (The Concepts in Secondary Mathematics Project reported by Hart [1981] has produced some interesting work in this area.)

The premature introduction of symbols to represent structures leads to pupils developing incorrect and inflexible schemas. Once a schema is established it tends to be firmly held, and pupils tend to alter their perception of contradicting concepts in order to fit them into their schema. One difficulty which many secondary pupils have in reading problems is that they construe the words so as to fit their firm schema rather than accept the intended meaning. This is one of the problems with the traditional model example and practice and theorem followed by rider methods of teaching mathematics which leads to externally imposed schema at too early a stage. This tends to encourage inflexibility and hence later a limited range of application of the schema. The method does, however, give those gifted pupils who can accommodate and change their schema an appreciation of the structure and a language in which to discern the form of mathematics.

The recent trend towards individualised learning methods, on the other hand, do give the pupils a broad base of low level concepts from which schema can be built. They allow the pupil to mark out the territory of a symbol by using it initially more as a temporary sign for a limited content, related to a short piece of work. These methods, however, seem to have difficulty in developing symbols relating to underlying structures. Because the pupils are using low level signs, they are not easily led to consider high level relationships. This absence of a symbolic language in which to recognise higher concepts leads the pupil to concentrate on easier low level concepts for which the language is available. The broad base of the triangle of mathematical knowledge which these methods create can be dissipated unless the pupils are also given the language and encouragement to build from this base.

The changes in content during the 1960's led to a considerable increase in the use of symbols; the introduction of set notation, functional notation, vectors, matrices, symbols for inverses, magnitudes, and logic. That this plethora of symbols did not cause any real disturbance might superficially seem a little surprising. The introduction of extra symbolism, however, serves to give the pupil more language in which to express and refine his ideas. Many of these new symbols were also operating at the level of signs, representing low level concepts

and distinguishing between ideas which were otherwise confused within the same symbol. One of the major problems which did become apparent was the insecurity of teachers with this symbolism, and this led to a pedantry which was out of step with the initial intentions of some reformers but which nevertheless came to be one of the characteristics of the changes. When a symbol is not securely understood, the edges of its meaning are avoided. There was a confusion, too, between the use of a symbol in the classroom for a single concept and the use of the symbol in more developed mathematics for a whole structural idea. This was enhanced by the teacher's own enjoyment in having mastered something new and wishing to pass onto the pupil immediately this whole concept of mathematical structure which had often escaped him (the teacher) in the past. The halo effect caused by this is unfortunately transitory. To mix the metaphors thoroughly, jumping from bandwagon to bandwagon can be exhilarating for teacher and pupil but is ultimately very tiring!

Whilst some features of these reforms will gradually disappear, some of the notational innovations will continue to prove advantageous. The contrast of the algebra of vectors and matrices with the algebra of number serves to help identify the more usual manipulations and encourage an appreciation of their structure. The availability of a symbol for magnitude can serve usefully to identify this particular idea from within more complex concepts (provided that it is used when required and not when it is superfluous). The idea of placeholders, solution sets, and function have not so far proved effective in the crucial problem of dealing with variables. The variety of concepts attached to, say, $y = 3x + 2$ needs a much more varied notation in the early stages. The confusion between when x and y are specific values (e.g., simultaneous equations) or representational values (drawing graphs or expressing a generalisation) or true variables (expressing abstract conceptual relationships) is present throughout mathematics and only sophisticated schema can really distinguish between them and accept their equivalencies. Many situations we present to students contain all three meanings at different stages within the same problem and the students certainly have difficulty and uncertainties as a result. The teacher, indeed, has subsumed these concepts into one schema needing one symbol, and since he does not need to differentiate he loses the facility.

The introduction of the ideas of functions and relations for use in different situations was an attempt to clarify this for the pupil, but the discrimination is only partly accomplished, and the general tendency to adopt only one or the other notation regardless of the problem concerned led to little overall improvement. More flexibility is certainly needed in the early stages of algebra, and less pedantry. Use could be made of boxes, circles, triangles, etc., when specific values are intended, and pupils should be encouraged to invent symbolism for unknown quantities and for representational situations such as generalised statements for patterns, e.g., sequences. The pupil's recognition of a need is often the best springboard for symbolism, and that symbolism must reflect that need. Mathematicians, indeed, use a great deal of implicit discriminants, such as using x, y, z for variables, a, b, c for coefficients, k, l, m for constants, and even using Greek letters, 'curly' letters, and so on. These distinctions are not readily discernable by the learner nor always conscious in the teacher and more distinct symbolisation is needed in the early stages, with the more usual conventions being allowed to grow slowly.

Symbols which Unify and Separate

One of the recurring problems is the use of a symbol on one hand to distinguish between concepts and on the other to unify concepts into more useful general schema which ignore irrelevancies. The result is that pupils cannot focus on either the woods or the trees. The value of symbols in developing simplifying structural schemas is very evident. The idea of differentiation as an operator leads to $(D^2 + 2D + 1)y = 0$ with an immediate recall of a known schema, and the possibility of extension to higher order. At a similar level of study the introduction of complex numbers in the form $re^{i\theta}$ can, and should, be dramatic. Indeed the variety of forms of complex numbers is also a good example of the use of symbols to distinguish between different facets of the same concept. Similarly at an earlier stage the use of different expressions— $16 = 2 \times 8 = 7 + 9 = 4^2 = 5^2 - 3^2 = \ldots\ldots$ —serve to emphasise different features. The expressions $16 = 1 + 15 = 2 + 14 = 3 + 13 = \ldots\ldots$ identify both different partitions and also a common feature. In introducing set language the need for a set to be well-defined is usually stressed, followed very soon by Venn diagrams in which the only specification is 'subsets of the Universal set.' Particularise, for example, the situations shown in Figure 2.

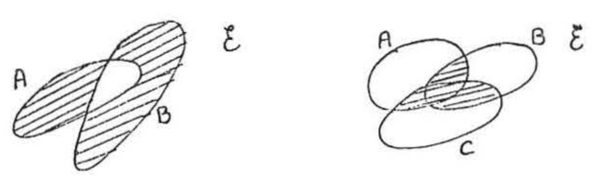

Figure 2.

The use of $a^{1/2}$ is an interesting situation in that it is at the same time the 'opposite' or 'inverse' of a^2 and also merely an extension of the exponential process $a^3\, a^2\, a^1 \ldots a^0 \ldots$. Consider the distinctions and similarities of the three statements:

A. $x + y = 7$, $3x + 2y = 15$ B. The lines $x + y = 7$ and $3x + 2y = 15$ and their intersection. C. $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$

In A, x and y are specific particular values whilst in B there are two different independent variables and two different dependent variables which (since they are in one plane) can take the same values simultaneously at the intersection. In C, x and y are characteristics of a single vector quantity. For the teacher with a well developed schema of ordered pairs there is value in using the same letters in all three situations. The pupil is likely to be in a situation similar to the pre-conservation era of Piagetian theory and unable to appreciate the constancies within these three examples. They therefore serve merely to confuse until 'the penny drops.' Unless the appreciation occurs reasonably early in the learning process this confusion passes past its initial usefulness (in altering the pupils' schemas into more useful ones based on higher level categorisations) into dismay and rejection.

The linear function $f(x) = mx + c$ (or $y = mx + c$) is another interesting example. To the teacher, conscious of many other functions, the role of m and c in determining the behaviour of the function is very clear. To the pupil this is hardly a linear function at all but many different functions, since the importance of linearity only arises in contrast to many non-linear functions. His concentration is solely on the many values of m and c and therefore each function is distinct and individual.

Nevertheless, without the use of similar notations the crucial structural categorisations may remain hidden. What is needed at school level are notations in which both similarities and differences are evident. At a higher level such notations are normal, for example d/dx and $\partial/\partial x$, $\int$ and $\oint$, log and ln x, sin x and sinh x. The need of mathematicians for this kind of clued notation has not been reflected in our school notations where the need is likely to be much greater.

Some Tentative Implications

The attachment of symbols to structural schemas rather than simple concepts would suggest that they come into play only in the latter stages of learning mathematics. This is related to the teaching feature stressed by Skemp in the use of one sign or symbol for one concept or schema *of the learner*. In the early learning of algebra, symbols are used not only for different concepts but also for different types of concept such as particulars (missing numbers), generalisations (extensions of pattern), and abstractions (expressions for structure or form). These are distinctive elements, which being distinctive need distinctive notations.

There has been a tendency for some years to use non-literal signs such as boxes in the early stages of the algebra of unknowns, and the introduction of the term 'placeholder' was indicative of this trend. In the early 1960's R. G. Davies made an interesting film of a lesson in which one group of children devise a relation between △ and □ to produce an answer ○. The other children by specifying trial numbers for △ and □ and being told the resultant ○ try to establish the relation. These introductions to algebra have never, however, succeeded in becoming more than trends. The arguments in this article suggest a much greater extension and development of their use. The introduction of literal signs brings with it a greater feeling of permanence, and it is essential that this permanence does not also produce rigidity, since even simple concepts must adapt and change as maturity and sophistication grow. The discussion has led to a plea for a greater range and variety of literal symbols in the early stages, which can both serve to distinguish and unify. The arbitrary and indiscriminate use of any letter in addition to the ubiquitous x does not in itself satisfy both these requirements, but a carefully thought-out development in which similar situations had similar but distinct notations is needed. One common example is the use of bold or italic letters for vectors, points, and magnitudes. In establishing the underlying structures of which algebra is the manifestation not nearly enough attention has usually been paid to the importance of having non-examples available to help establish characteristic qualities. In particular, the concentration on an analysis of linear functions in most school syllabuses is attempted without sufficient attention to establishing the concept of linear functions. Indeed, the idea of operators and function machines rather than more general functions would seem much more pertinent in school mathematics, since the pupil has a much greater variety of experience upon which to draw. This is also reflected, perhaps, in the complaints of teachers of other subjects to whom the higher level idea of a function seems hardly as relevant. They desire the ability to manipulate single operators in sequence, whereas the concept of a function is an appreciation of the results of combining multiple operations.

This approach leads to a stress early in the course on topics similar to the traditional transformation of formulae but placed in a less algebraic setting by the use of diagrams and flow charts; e.g., such

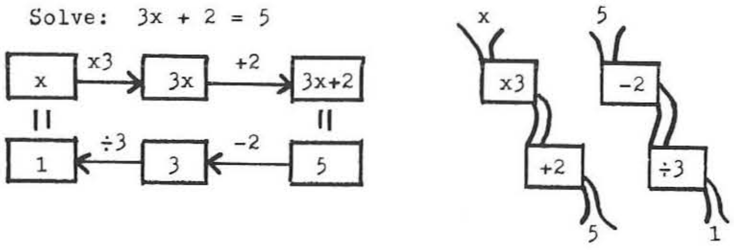

Figure 3.

representation as shown in Figure 3. This approach could build upon some of the work contained in some primary school syllabuses — (see Fletcher 1971). This use of operators leads naturally to the use of functional notation for combinations of operators at a later school stage. In the early stages of studying functions, pedantic mathematical distinctions of notation should not be demanded of the pupils, even though the teacher may well choose his own notation for a situation in anticipation of more advanced criteria. As the pupils' schemas develop so the notation can be refined; all that is necessary is the availability of suitable notations when the need for these refinements arise.

It is likely that we teachers will find it difficult to alter our own notational schemas to fit the pupils' needs. Just as teachers deplore the inability of their pupils to solve $a + bx + cx^2 = 0$ so we may have difficulty in making such simple but useful adaptations as using $Ax^2 + Bx + C = 0$ or $f(x) = Mx + C$. Such a change may seem trivial to the teacher who intuitively distinguishes between coefficients and variables (unknowns?). The change in size of notations, however, suggest such a distinction much more clearly to the pupil.

The importance of symbolism in mathematics is indisputable, but we have little research evidence on the learning of mathematical symbols. There is a great deal of expertise known to experienced teachers. Much is accomplished by hand-waving and individual ad-hoc symbols, but this needs to be externalised and theorised so as to become available to the whole community of mathematics teachers and to help overcome deficiencies of both syllabuses and texts. The lack of teachers with a secure and sound training in mathematics is unlikely to be overcome quickly, and without security there is no flexibility. It is therefore increasingly urgent that advice which rests upon a systematic and realistic theory of learning is made available.

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